Theory of Relativity

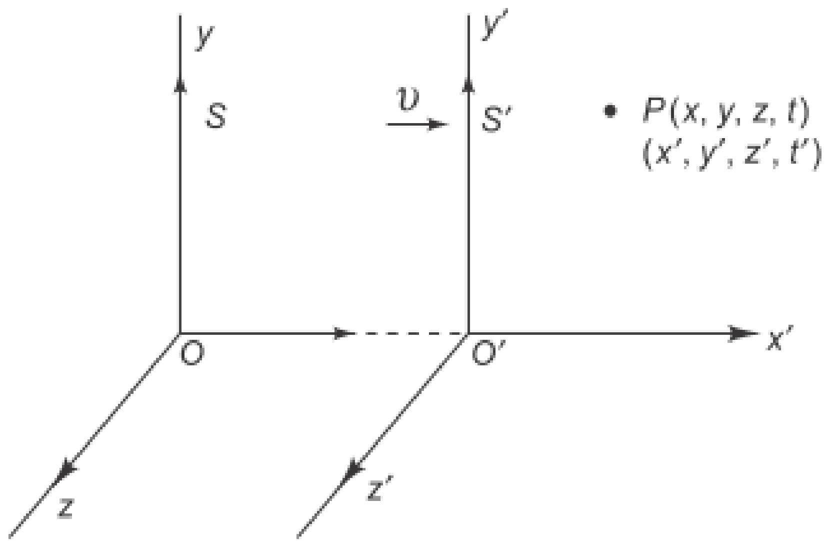
UNIT I
Relativistic Mechanics
Lecture-3

हमे हर वक्त ये एहसास दामनगीर रहता है

पडे है ढेर सारे काम और मोहलत जरा सी है

LORENTZ TRANSFORMATION EQUATIONS

- Using the postulates of special theory of relativity Lorentz derive the real relativistic transformation equations known as Lorentz transformation equations of space and time



LORENTZ TRANSFORMATION EQUATIONS CONTD…

- The measurement in the x-direction made in frame S is proportional to that made in frame S ’

$$x' = k(x - vt)$$

where k is the factor independent of x and t but may be the function of $\vec{v}$

- On the basis of first postulate of special theory of relativity, we can write the same equation for x which can be determined in terms of x’ and t’. Thus, we can write

$$x = k(x' - (-vt'))$$

LORENTZ TRANSFORMATION EQUATIONS CONTD…

- Using the value of x’ we can find the value of t’ in terms of t, v and x

$$t' = \frac{(1-k^2)}{kv}x + kt$$

- Let a light signal be given at the origin O at time t = t’= 0. The distance travelled by the signal in frames S and S’ can be given as follows:

In frame S,

$$x = ct$$

and in frame S',

$$x' = ct'$$

LORENTZ TRANSFORMATION EQUATIONS CONTD…

- Using the values of x’ and t’ we can write

$$x' = c\left\{\frac{(1-k^2)x}{k\upsilon} + kt\right\}$$

$$= \frac{c(1-k^2)x}{k\upsilon} + ckt$$

$$k(x-\upsilon t) = \frac{c(1-k^2)x}{k\upsilon} + ckt$$

- Simplifying above equation and putting the value of x we get

$$k = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

LORENTZ TRANSFORMATION EQUATIONS CONTD…

- Putting the value of k in following equation

$$t' = \frac{(1-k^2)}{kv}x + kt$$

- We get

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x' = \frac{(x-vt)}{\sqrt{1-\frac{v^2}{c^2}}},\ y' = y,\ z' = z, \text{ and } \quad t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Inverse Lorentz Transformation Equations

- In this transformation frame S' is static and S is moving in backward direction

- Thus the Inverse Lorentz transformations can be obtained by changing non dashed coordinates to dashed coordinate and replacing v by –v in the Lorentz transformation equations.

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Consequences of Lorentz Transformation Equations

Consequences of Lorentz Transformation Equations

- Length Contraction
- Time Dilation

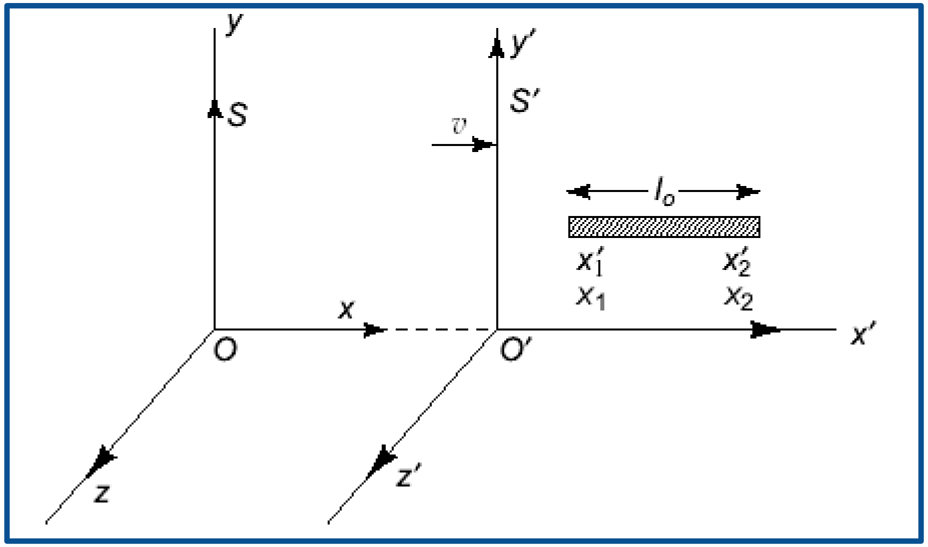
Length Contraction

- **Let us consider a rod of proper length** ***l_0 placed in a moving frame of reference and l is the length of rod observed by the observer being in stationary frame.***

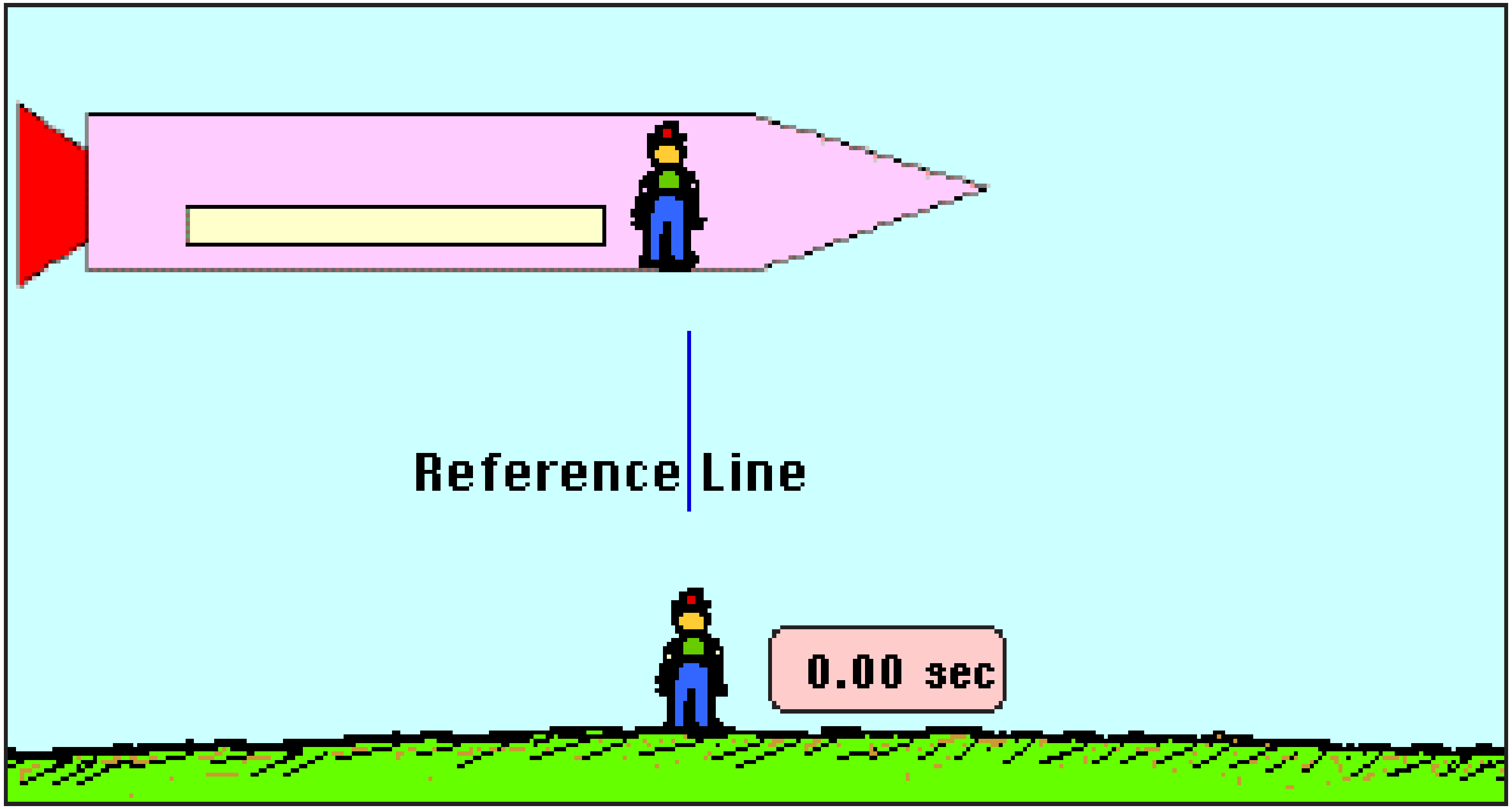
$$l_0 = x'_2 - x'_1$$

$$l = x_2 - x_1$$

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$



Length Contraction



man on rocket

time=t
L = vt

Time = t'=t/k

Length (L')= vt'=vt/k= L/k= $L\sqrt{1-\frac{v^2}{c^2}}$

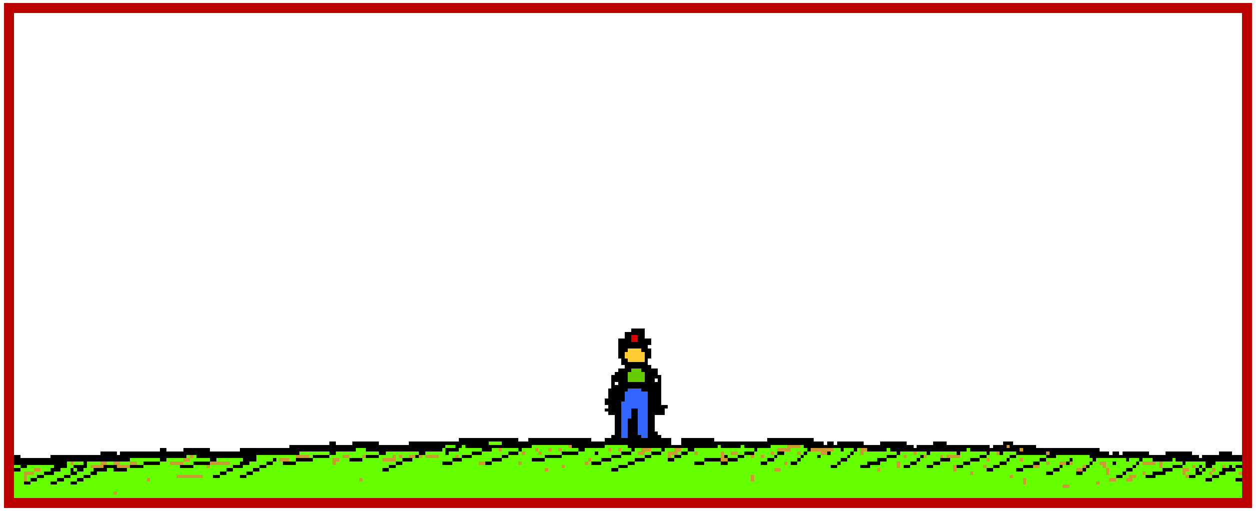
Shorter!

Moving objects appear shorter

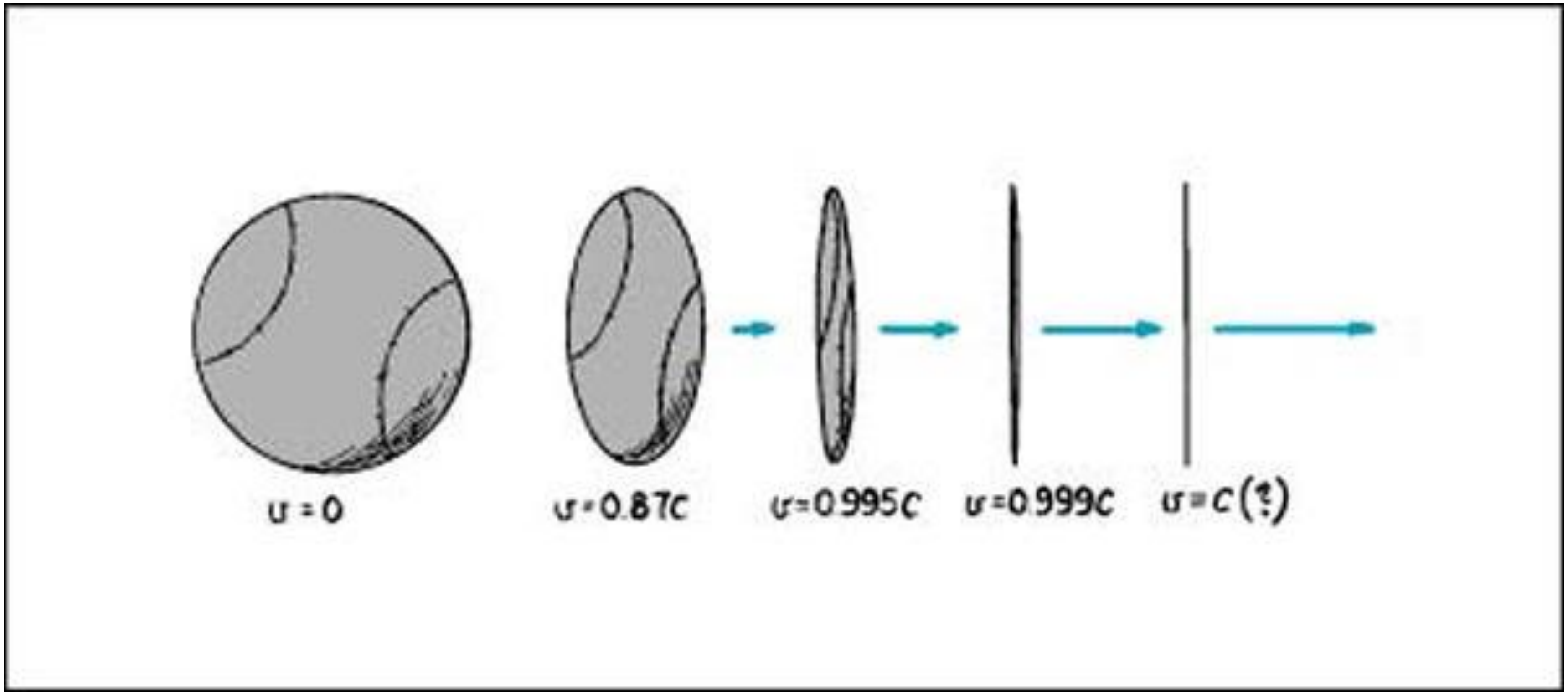
$$L = L_o/\kappa$$

Length measured when object is at rest

$\kappa > 1 \Rightarrow L < L_o$

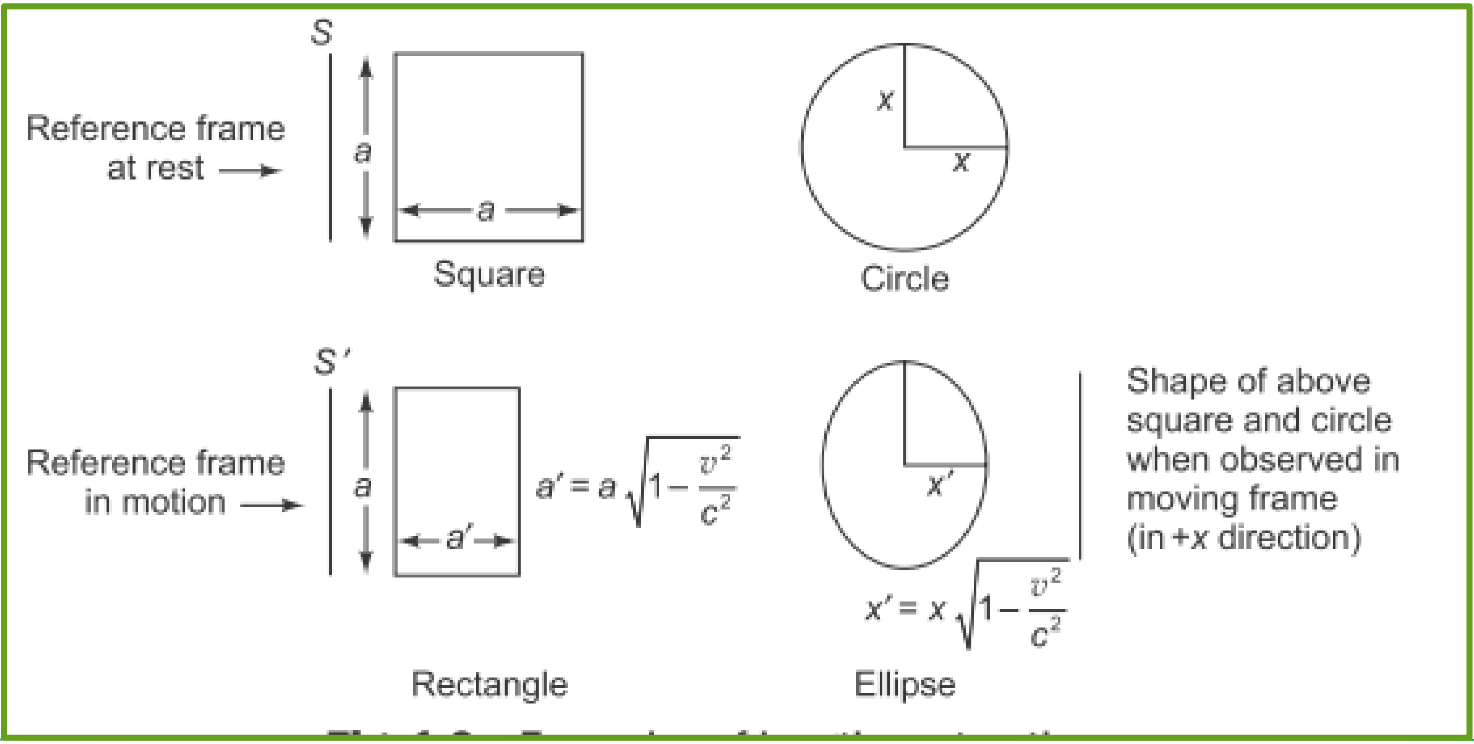


Length Contraction



Examples of Length contraction

- Due to the phenomenon of length contraction, a circle and a square in one frame of reference (stationary) appear to the observer in the other frame (moving) as ellipse and rectangle, respectively, as shown in following figure



Time Dilation

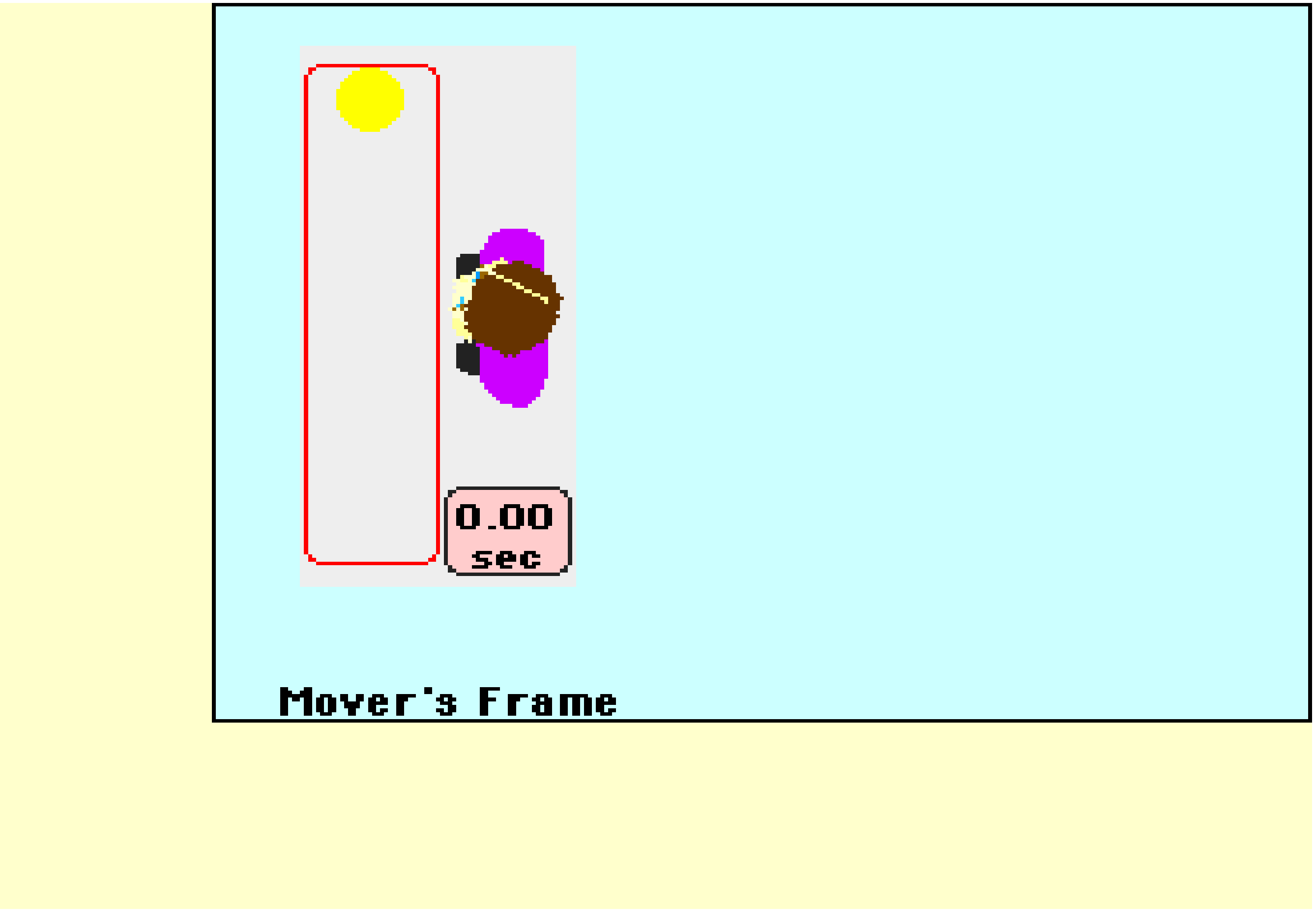
- According to the time dilation, if a clock at rest in the frame *S' measures the times* t'_1 *and* t'_2 *of two events occurring at a fixed position* x' *in this frame, then the time interval between these events is known as proper time and is given as* $\Delta t' = t'_2 - t'_1$

- If t_1 *and* t_2 *are the times of same events recorded by a clock at rest in the frame S, then* $\Delta t = t_2 - t_1$

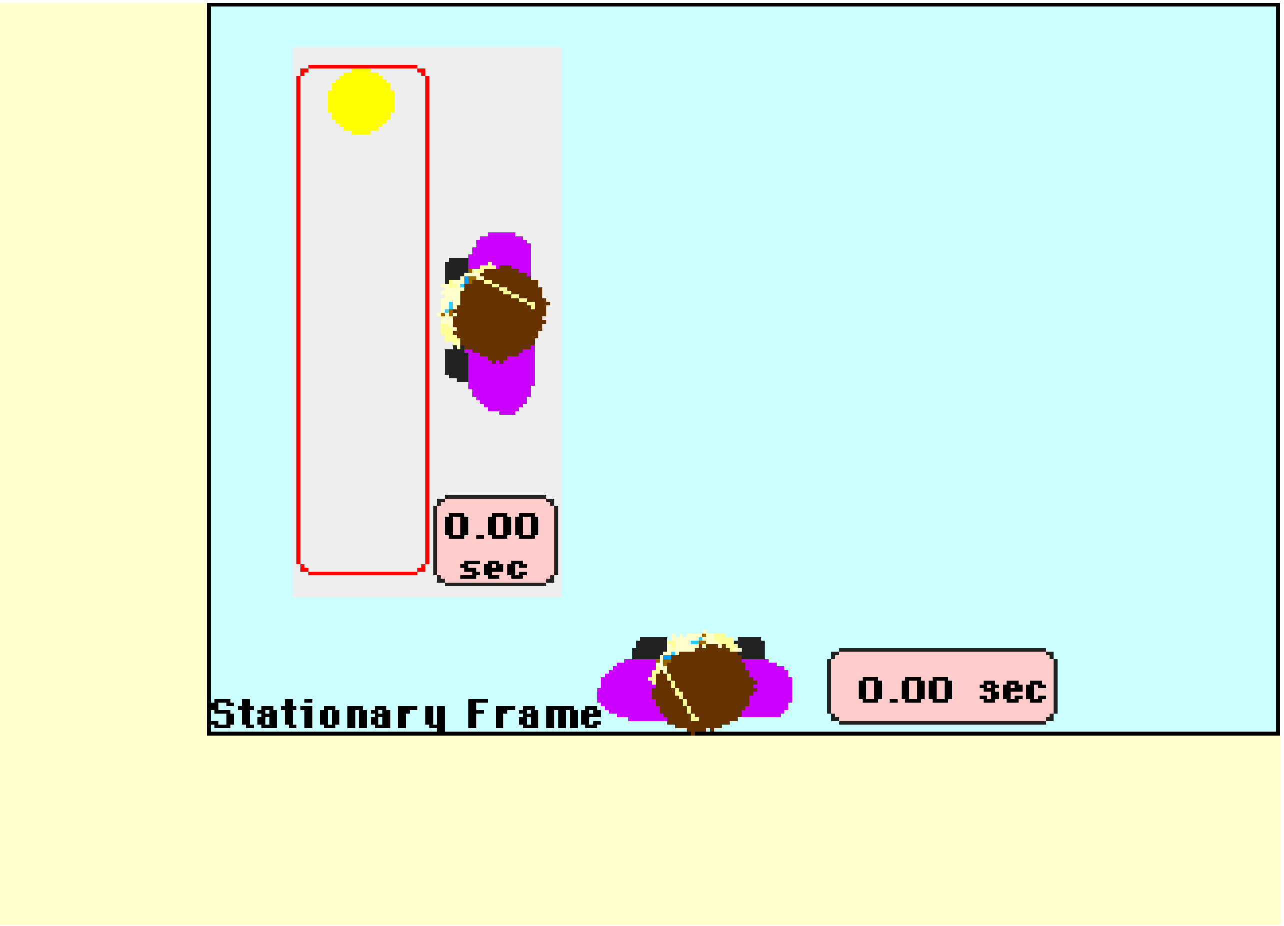
- *Now the relation between* Δt *and* $\Delta t'$ *can be given as*

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Light Clock

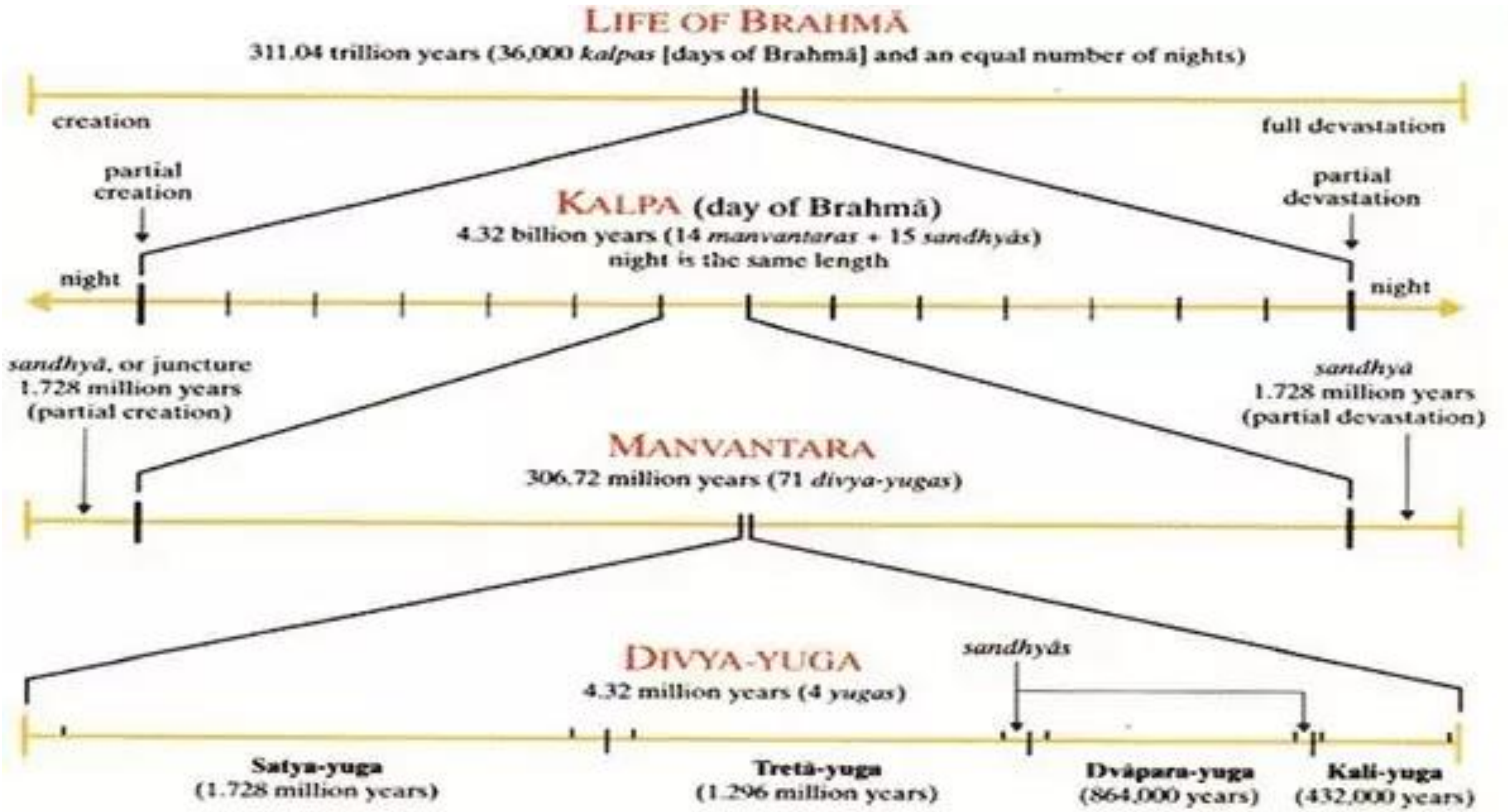


Sees from ground



Mythical Evidence (Age of Brahmaji)

- *1 DAY OF BRAHMA= 14 MANU's + 15 Junction points= 14 x 71 x 4,320,000 years+ 15 x 1,728,000 = 4,294,080,000+25,920,000 =4,320,000,000 years (4.32 billion)*



Experimental verification of time dilation

- Experimentally, it is verified that the time dilation is a real effect.
- It can be justified by taking the example of meson decay.
- A μ-meson is an elementary particle whose mean lifetime is 2.2 x 10^{-6} s in the frame in which it is at rest.
- Such meson particles have their speed 2.994 x 10^8 m/s.
- These particles are created 8–10 km above the surface of the earth in the atmosphere by fast cosmic ray particles.

Experimental verification of time dilation

- When these particles travel with the velocity 2.994 x 10^8 m/s, then in time 2.2 x 10^{-6} s, they can travel a distance

$$2.994 \times 10^8 \times 2.2 \times 10^{-6} = 6.5868 \times 10^2 \text{ m}$$
$$= 658 \text{ m}$$

Using the concept of time dilation for meson particles, the lifetime will be given as

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.2 \times 10^{-6}}{\sqrt{1 - \frac{(2.994 \times 10^8)^2}{(3 \times 10^8)^2}}} = 34.8 \times 10^{-6} \text{ s}$$

- With this lifetime (34.8 x 10^{-6} s), the μ -particles can travel a distance of 2.994 x 10^8 m/s x 34.8 x 10^{-6} s = 10.42 km.

Conceptual Questions

Show that $x^2 + y^2 + z^2 - c^2 t^2$ is invariant under the Lorentz transformation.

or

Show that the space–time interval between two events remains invariant under the Lorentz transformation.

Solution

We have to show that $x^2 + y^2 + z^2 - c^2t^2$ remains invariant, i.e., the form of expression remains as such in the inertial frames S and S'.

From the inverse Lorentz transformation, we know that

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \; y = y', \; z = z', \text{ and } \; t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Using these values of coordinates in the given expression, we get

$$\left(\frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2 + y'^2 + z'^2 - c^2 \left[\frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\right]^2$$

$$= y'^2 + z'^2 - \frac{1}{\left(1 - \frac{v^2}{c^2}\right)} \left[c^2 t'^2 + \frac{v^2 x'^2}{c^2} + 2vx't' - x'^2 - 2vt'x' - v^2 t'^2\right]$$

$$= y'^2 + z'^2 - \frac{1}{\left(1 - \frac{v^2}{c^2}\right)} \left[-x'^2 + c^2 t'^2\right]\left(1 - \frac{v^2}{c^2}\right)$$

$$= x'^2 + y'^2 + z'^2 - c^2 t'^2$$

Conceptual Questions Contd…

Show that the circle $x^2 + y^2 = a^2$ in frame S appears to be an ellipse in frame S' that is moving with velocity υ relative to S.

Solution

In frame S, the equation of a circle in the stationary frame is

$$x^2 + y^2 = a^2$$

In frame S',

$$x' = x\sqrt{1 - \frac{\upsilon^2}{c^2}} \qquad \text{(Using length contraction)}$$

and $$y' = y \qquad \text{(because } S' \text{ is moving along +ve } x\text{-axis)}$$

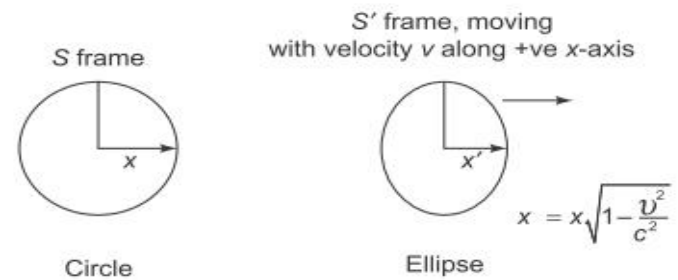


Fig. Shape of a circle in moving frame

Substituting these values in the equation of circle, we get

$$\frac{x'^2}{a^2\left(1 - \frac{\upsilon^2}{c^2}\right)} + \frac{y'^2}{a^2} = 1$$

or $$\frac{x'^2}{b^2} + \frac{y'^2}{a^2} = 1, \text{ where } b^2 = a^2\left(1 - \frac{\upsilon^2}{c^2}\right)$$

This is the equation of an ellipse.

Conceptual Questions Contd…

A clock measures the proper time. With what velocity should it travel relative to an observer so that it appears to go slow by 30 s in 12 h.

Solution

Let t_0 be the proper time and t be the apparent time for the moving frame of reference. Here, it is given that the clock appears to go slow by 30 s in the moving frame. Hence, 12 h will appear as 12 h + 30 s in the frame of reference where the clock appears to move.

Now, from the expression of time dilation we can write

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$12 \times 60 \times 60 + 30 = \frac{12 \times 60 \times 60}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or $$\sqrt{1 - \frac{v^2}{c^2}} = \frac{43200}{43230}$$

$$= 0.999306$$

or $$1 - \frac{v^2}{c^2} = 0.998612$$

Assignment based on what we learnt in this lecture ?

- Define length contraction and time dilation.
- Obtain the expression for the length contraction and time dilation.
- With suitable example show that time dilation is a real effect.
- Explain time dilation using the example of twin paradox.
- A clock keeps correct time. With what speed should it be moved relative to an observer so that it may appear to loose 4 min in 24 h.

Numerical Questions

- The mean lifetime of a μ-meson when it is at rest is 2.2 x 10^{-6} s. Calculate the average distance it will travel in vacuum before decay if its velocity is 0.8 c.
- A rocketship is 100 m long on the ground. When it is in flight, its length is 99 m to an observer on the ground. What is its speed?
- Calculate the percentage contraction in the length of rod in a frame of reference moving with velocity 0.8c in the direction parallel to its length.