Control Systems

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Unit-II

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Transfer Function from State Space Model

We know the state space model of a Linear Time-Invariant (LTI) system is -

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

Apply Laplace Transform on both sides of the state equation.

$$sX(s) = AX(s) + BU(s)$$

$$\Rightarrow (sI - A)X(s) = BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1}BU(s)$$

Apply Laplace Transform on both sides of the output equation.

$$Y(s) = CX(s) + DU(s)$$

Substitute, X(s) value in the above equation.

$$\Rightarrow Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$\Rightarrow Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

The above equation represents the transfer function of the system. So, we can calculate the transfer function of the system by using this formula for the system represented in the state space model.

Note – When $D = [0]$, the transfer function will be

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Example

Let us calculate the transfer function of the system represented in the state space model as,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad and \quad D = [0]$$

The formula for the transfer function when $D = [0]$ is -

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Substitute, A, B & C matrices in the above equation.

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}}{(s+1)s - 1(-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2 + s + 1} = \frac{1}{s^2 + s + 1}$$

Therefore, the transfer function of the system for the given state space model is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

State Transition Matrix and its Properties

If the system is having initial conditions, then it will produce an output. Since, this output is present even in the absence of input, it is called **zero input response** $x_{ZIR}(t)$. Mathematically, we can write it as,

$$x_{ZIR}(t) = e^{At}X(0) = L^{-1}\left\{[sI - A]^{-1}X(0)\right\}$$

From the above relation, we can write the state transition matrix $\phi(t)$ as

$$\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}$$

So, the zero input response can be obtained by multiplying the state transition matrix $\phi(t)$ with the initial conditions matrix.

Following are the properties of the state transition matrix.

- If $t = 0$, then state transition matrix will be equal to an Identity matrix.

$$\phi(0) = I$$

- Inverse of state transition matrix will be same as that of state transition matrix just by replcing 't' by '-t'.

$$\phi^{-1}(t) = \phi(-t)$$

- If $t = t_1 + t_2$, then the corresponding state transition matrix is equal to the multiplication of the two state transition matrices at $t = t_1$ and $t = t_2$.

$$\phi(t_1 + t_2) = \phi(t_1)\phi(t_2)$$

THE TRANSFER FUNCTION FROM THE STATE EQUATION

The transfer function of a single input-single output (SISO) system can be obtained from the state variable equations.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where y is the single output and u is the single input. The Laplace transform of the equations

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s)$$

where B is an nx1 matrix, since u is a single input. We do not include initial conditions, since we seek the transfer function. Reordering the equation

$$[sI - A]X(s) = BU(s)$$

$$X(s) = [sI - A]^{-1}BU(s) = \phi(s)BU(s)$$

$$Y(s) = C\phi(s)BU(s)$$

Therefore, the transfer function G(s)=Y(s)/U(s) is

$$G(s) = C\phi(s)B$$

Example:

Determine the transfer function G(s)=Y(s)/U(s) for the RLC circuit as described by the state differential function

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

$$[sI-A]=\begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s+\frac{R}{L} \end{bmatrix}$$

$$\phi(s)=[sI-A]^{-1}=\frac{1}{\Delta(s)}\begin{bmatrix} s+\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

$$\Delta(s)=s^2+\frac{R}{L}s+\frac{1}{LC}$$

Then the transfer function is

$$G(s)=\begin{bmatrix}0 & R\end{bmatrix}\begin{bmatrix} \frac{s+\frac{R}{L}}{\Delta(s)} & -\frac{1}{C\Delta(s)} \\ \frac{1}{L\Delta(s)} & \frac{s}{\Delta(s)} \end{bmatrix}\begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$G(s)=\frac{R/LC}{\Delta(s)}=\frac{R/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Example: Consider the third-order system

$$G(s) = \frac{Y(s)}{R(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$

We can obtain a state-space representation using the **ss** function. The state-space representation of the system given by G(s) is

$$A = \begin{bmatrix} -8 & -4 & -1.5 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0.75 \end{bmatrix} \text{and } D = \begin{bmatrix} 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -8 & -4 & -1.5 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

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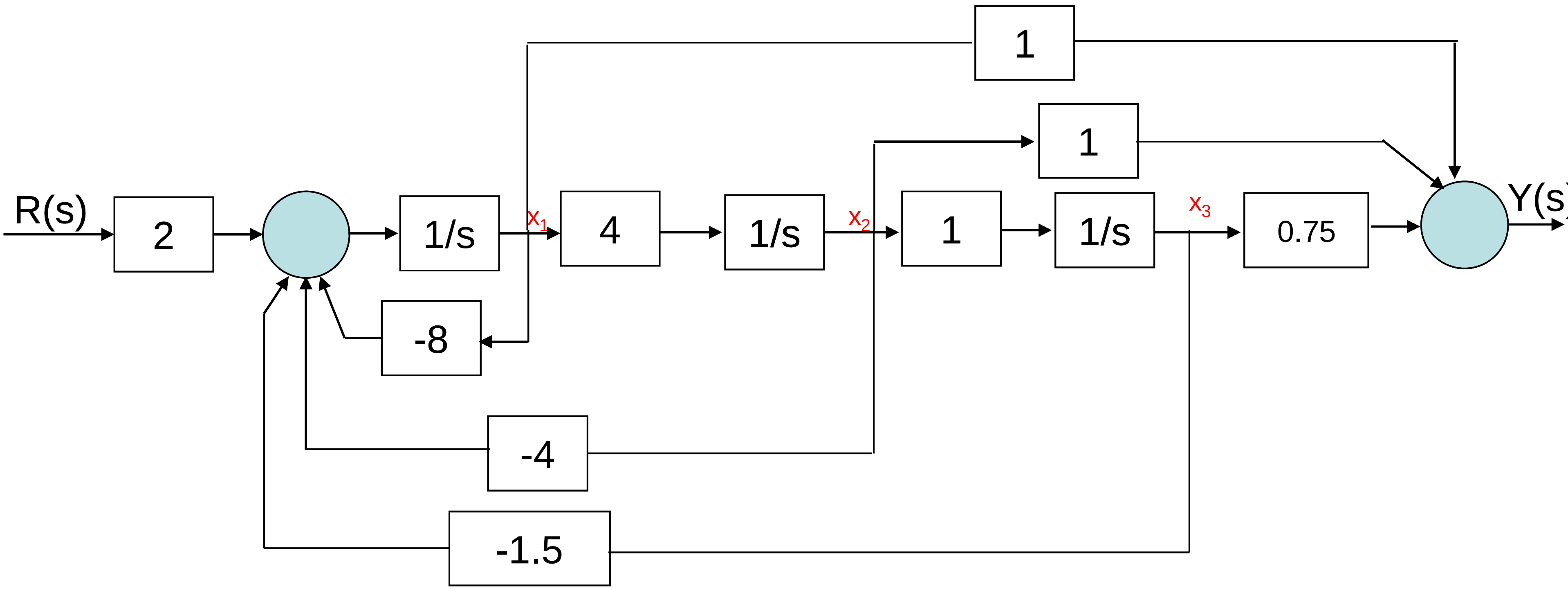

Block diagram with x_1 defined as the leftmost state variable.

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$

For the RLC network, the state-space representation is given as:

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{and } D = \begin{bmatrix} 0 \end{bmatrix}$$

The initial conditions are $x_1(0)=x_2(0)=1$ and the input $u(t)=0$.