Subject Name-
ADVANCE QUANTUM MECHANICS

Subject Code- MPM-221

Teacher Name

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Syllabus??

MPM-221: ADVANCE QUANTUM MECHANICS **Credit 04 (3-1-0)**

Unit I: Formulation of Relativistic Quantum Theory

Relativistic Notations, The Klein-Gordon equation, Physical interpretation, Probability current density & Inadequacy of Klein-Gordon equation, Dirac relativistic equation & Mathematical formulation, α and β matrices and related algebra, Properties of four matrices α and β, Matrix representation of $\alpha_i^{'s}$ and β, True continuity equation and interpretation.

Unit II: Covariance of Dirac Equation

Covariant form of Dirac equation, Dirac gamma (γ) matrices, Representation and properties, Trace identities, fifth gamma matrix γ^5, Solution of Dirac equation for free particle (Plane wave solution), Dirac spinor, Helicity operator, Explicit form, Negative energy states

Unit III: Field Quantization

Introduction to quantum field theory, Lagrangian field theory, Euler–Lagrange equations, Hamiltonian formalism, Quantized Lagrangian field theory, Canonical commutation relations, The Klein-Gordon field, Second quantization, Hamiltonian and Momentum, Normal ordering, Fock space, The complex Klein-Gordan field: complex scalar field

Unit IV: Approximate Methods

Time independent perturbation theory, The Variational method, Estimation of ground state energy, The Wentzel-Kramers-Brillouin (WKB) method, Validity of the WKB approximation, Time-Dependent Perturbation theory, Transition probability, Fermi-Golden Rule

Books & References:

1: Advance Quantum Mechanics by J. J. Sakurai (Pearson Education India)

2: Relativistic Quantum Mechanics by James D. Bjorken and Sidney D. Drell (McGraw-Hill Book Company; New York, 1964).

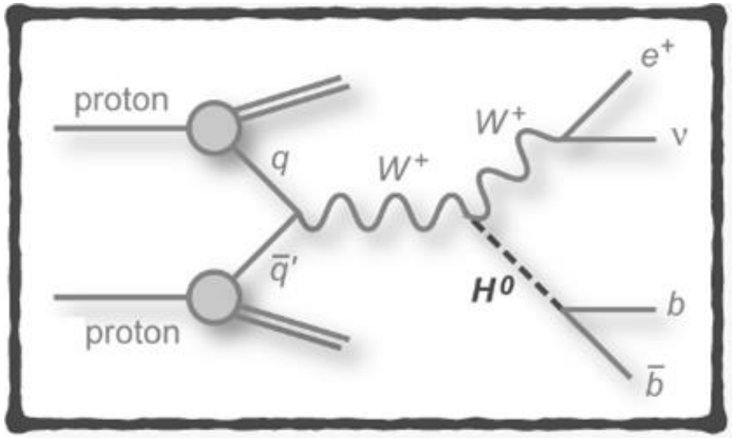
3: An Introduction to Relativistic Quantum Field Theory by S.S. Schweber (Harper & Row, New York, 1961).

4: Quantum Field Theory by F. Mandl & G. Shaw (John Wiley and Sons Ltd, 1984)

5: A First Book of Quantum Field Theory by A. Lahiri & P.B. Pal (Narosa Publishing House, New Delhi, 2000)

Session 2020-21

Lectures of Unit- III



Unit III: Field Quantization

Introduction to quantum field theory

Unit - III

1

Quantum Field Theory (QFT)

What is Quantum Field Theory? wiki

Quantum Field Theory :- as its name implies, it is the quantization of classical field, the most familiar example of which is the electromagnetic field. In standard quantum mechanics, we take the classical degrees of freedom and promote them to operators acting on a Hilbert space. The rules for quantizing a field are no different. Thus the basic degrees of freedom in quantum field theory is operator valued functions of space and time. This means that we are dealing with an infinite no. of degrees of freedom - at least one for every point in space.

For any relativistic system QFT is a necessary. But it is also a very useful tool in non-relativistic systems with many particles. QFT has had a major impact in condensed matter, high-energy physics, cosmology, quantum gravity and pure mathematics. It is literally the language in which the laws of Nature are

Unit III: Field Quantization

Classical field theory

Classical Field Theory :—

The dynamics of Fields :—

A field is a quantity defined at every point of space and time $(\vec{x}, t)$. While classical mechanics deals with a finite number of generalized coordinate $q_r(t)$, indexed by a label r, in field theory we are interested in the dynamics of fields

$$\phi_r(\vec{x}, t) \longrightarrow (1)$$

where both r and $\vec{x}$ are considered as labels. Thus we are dealing with a system with an infinite number of degrees of freedom — at least one for each point $\vec{x}$ in space.

An example: The Electromagnetic Fields:—

The most familiar examples of field from classical physics are the electric and magnetic fields, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$. Both of these are spatial 3-vectors. We can derive these two 3-vectors from a single 4 component field.

$$A^{\mu}(\vec{x}, t) = (\phi, \vec{A}) \quad ; \quad \mu = 0,1,2,3 \longrightarrow (2)$$

This shows that field is a vector in spacetime.

The electric & magnetic fields are given by—

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A} \longrightarrow (3)$$

Lagrangian field theory

Classical Lagrangian Field Theory:-

We consider a system which requires several fields $\phi_r(x), r=1,2,\ldots,N$ as a characterizing specifying character of system (field), taken as field variable on each point of space Ω at x. The index r may label components of the same field or it may refer to different independent field

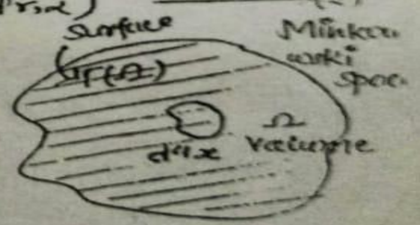
Now we restrict ourselves to theories which can be derived by variational principle from an action integral involving a Lagrangian density -

$$\mathcal{L} = \mathcal{L}(\phi_r, \phi_{r,\alpha}) \quad \text{------} \quad (1)$$

$$\text{where,} \quad \phi_{r,\alpha} = \partial_\alpha \phi_r = \frac{\partial \phi_r}{\partial x^\alpha}$$

We define the action integral $S(\Omega)$ for an arbitrary region Ω of the four dimensional space-time continuum by -

$$S(\Omega) = \int_\Omega d^4x \, \mathcal{L}(\phi_r, \phi_{r,\alpha}) \longrightarrow (2)$$



For an arbitrary region Ω, we consider variation of the fields,

$$\phi_r(x) \longrightarrow \phi_r(x) + \delta\phi_r(x) \longrightarrow (3)$$

which vanish on the surface $\Gamma(\Omega)$ bounding the region Ω.

$$\delta\phi_r(x) = 0 \quad \text{on } \Gamma(\Omega) \qquad (4)$$

The fields ϕ_r may be real or complex. In the case of complex field $\phi(x)$, the fields $\phi(x)$ and $\phi^*(x)$ are treated as two independent fields. Alternatively, a complex field $\phi(x)$ can be decomposed into a pair of real fields, which are then treated as independent fields.

For an arbitrary region and the variation, the action has a stationary value, i.e.

$$\delta S(\Omega) = 0 \longrightarrow (5)$$

From equation (2), we get

$$\delta S(\Omega) = \int_\Omega d^4x \, \delta[\mathcal{L}(\phi_r, \phi_{r,\alpha})]$$

$$= \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_r} \delta\phi_r + \frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \delta\phi_{r,\alpha} \right\}$$

$\because$

$$\delta\phi_{r,\alpha} = \frac{\partial}{\partial x^\alpha} \delta\phi_r$$

$\therefore$

$$\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \left(\frac{\partial}{\partial x^\alpha} \delta\phi_r \right) = \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \delta\phi_r \right) - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \right) \delta\phi_r$$

(using partial Integration)

Hence,

$$\delta S(\Omega) = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \right) \right] \delta\phi_r(x) + \int_\Omega d^4x \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \delta\phi_r(x) \right) \longrightarrow (6)$$

The last term in eqn (6) can be converted into a surface integral over the surface $\Gamma(\Omega)$ using Gauss's divergence theorem in

Lagrangian field theory

four dimensions.

$$\int_\Omega d^4x \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \delta\phi_r(x)\right) = \int_\Gamma dS \frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \delta\phi_r(x)$$

$$= 0 \quad (\text{Since } \delta\phi_r = 0 \text{ on } \Gamma)$$

$\therefore \delta S(\Omega) = 0$ Thus for arbitrary Ω finally we have-

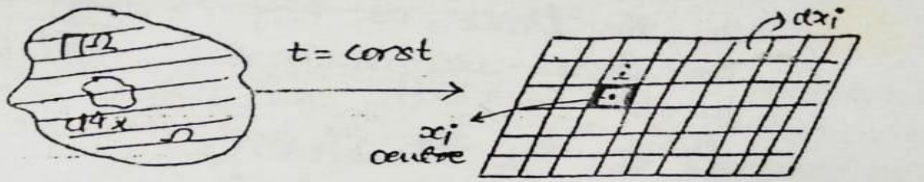
$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\right) = 0}, \; r = 1, 2, --N \longrightarrow (7)$$

These are the equations of motion of fields (The Euler - Lagrange equations)

We are dealing with a system with a continuous infinite number of degrees of freedom, corresponding to the values of the fields ϕ_r, considered as functions of time, at each point of space x. We shall again approximately the system by one having a countable number of degrees of freedom and ultimately go to the continuum limit.

Consider the system at fixed instant of time t and decomposes the three-dimensional space i.e. the flat-space-like surface t = const., into small cells of equal volume δx_i, labelled by the index $i = 1, 2, \ldots$ We approximate the values of the fields within each cell by their values at the centre of the cell $x = x_i$.

Lagrangian field theory



Flat-space-like Euclidean surface (3 dim)

The system is now described by the discrete set of generalized coordinates –

$$q_{ri}(t) \equiv \phi_r(i,t) \equiv \phi_r(x_i,t) \longrightarrow (8)$$
$$r = 1, 2, \dots, N \;;\; i = 1, 2, \dots$$

which are the values of the fields at the discrete lattice sites x_i. If we also replace the spatial derivatives of the fields by their difference coefficients between neighbouring sites, we can write the Lagrangian of the discrete system as –

$$L(t) = \sum_i \delta x_i \, \mathcal{L}_i\big(\phi_r(i,t), \dot{\phi}_r(i,t), \phi_r(i',t)\big) \longrightarrow (9)$$

Hamiltonian Formalism:

We define momenta conjugate to q_{ri} in the usual way as –

$$p_{ri}(t) = \frac{\partial L}{\partial \dot{q}_{ri}} \equiv \frac{\partial L}{\partial \dot{\phi}_r(i,t)} \equiv \pi_r(i,t)\,\delta x_i \longrightarrow (10)$$

$$\text{where } \pi_r(i,t) = \frac{\partial \mathcal{L}_i}{\partial \dot{\phi}_r(i,t)} \longrightarrow (11)$$

The Hamiltonian of the discrete system is given by –

$$H = \sum_i p_{ri}\dot{q}_{ri} - L$$

$$= \sum_i \delta x_i \{\pi_r(i,t)\dot{\phi}_r(i,t) - L_i\} \longrightarrow (12)$$

Taking the limit $\delta x_i \to 0$ i.e. letting the cell size and the lattice spacing shrink to zero, we define the fields conjugate to $\phi_r(x)$ as –

$$\pi_r(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \longrightarrow (13)$$

In the limit $\delta x_i \to 0$, $\pi_r(i,t) \to \pi_r(x_i,t)$ and the discrete Lagrangian and Hamilton functions (9) and (12) becomes –

$$L(t) = \int d^3x\, \mathcal{L}(\phi_r, \phi_{r,\alpha}) \longrightarrow (14)$$

and

$$H = \int d^3x\, \mathcal{H}(x) \longrightarrow (15)$$

where the Hamiltonian density $\mathcal{H}(x)$ is defined by –

$$\mathcal{H}(x) = \pi_r(x)\dot{\phi}_r(x) - \mathcal{L}(\phi_r, \phi_{r,\alpha}) \longrightarrow (16)$$

Integrations in (14) & (15) are all over all space at time t.

$$P^j = \int d^3x\, \pi_r(x) \frac{\partial \phi_r(x)}{\partial x^j} \longrightarrow (16')$$

Example :– Consider the Lagrangian density –

$$\mathcal{L} = \frac{1}{2}(\phi_{,\alpha}\phi^{,\alpha} - \mu^2\phi^2) \longrightarrow (17)$$

for a single real field $\phi(x)$ with μ a constant, which has the dimension of $(\text{length})^{-1}$.

The equation of motion of this field is the Klein-Gordon equation

$$(\Box + \mu^2)\,\phi(x) = 0 \qquad \longrightarrow (18)$$

The conjugate field is –

$$\pi(x) = \frac{1}{c^2}\,\dot{\phi}(x) \qquad \longrightarrow (19)$$

and the Hamiltonian density is

$$\mathcal{H}(x) = \frac{1}{2}\left[c^2\pi^2(x) + (\nabla\phi)^2 + \mu^2\phi^2\right] \qquad \longrightarrow (20).$$

<u>Quantized Lagrangian Field Theory</u>:–

Now it is easy to go from the classical to the quantum field theory by interpreting the conjugate coordinates and momenta of the the discrete lattice approximation, equation (8) and (10), as Heisenberg operators, and subjecting these to the usual canonical commutation relations:

$$\left.\begin{aligned} [\phi_r(j,t), \pi_s(j',t)] &= i\hbar\,\frac{\delta_{rs}\,\delta_{jj'}}{\delta x_j} \\ [\phi_r(j,t), \phi_s(j',t)] &= [\pi_r(j,t), \pi_s(j',t)] = 0 \end{aligned}\right\} \rightarrow (21)$$

let the lattice spacing go to zero, then (21) becomes –

$$\left.\begin{aligned} [\phi_r(\bar{x},t), \pi_s(\bar{x}',t)] &= i\hbar\,\delta_{rs}(\bar{x}-\bar{x}') \\ [\phi_r(\bar{x},t), \phi_s(\bar{x}',t)] &= [\pi_r(\bar{x},t), \pi_s(\bar{x}',t)] = 0 \end{aligned}\right\} \rightarrow (22)$$

Deriving Equation of Motion with Euler-Lagrange's method

Deriving eqn of motion with Euler - Lagrange's method :-

if Lagrangian density is given as,

$$\mathcal{L} = \frac{1}{2}[\phi_{,\mu}\phi^{,\mu} - \mu^2\phi^2)$$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi)(\partial^\mu \phi) - \mu^2\phi^2\right] \to (1)$$

$$\because \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \ \& \ \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

$$= \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x^\mu}\right)\left(\frac{\partial \phi}{\partial x_\mu}\right) - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x^0} + \frac{\partial \phi}{\partial x^1} + \frac{\partial \phi}{\partial x^2} + \frac{\partial \phi}{\partial x^3}\right)\left(\frac{\partial \phi}{\partial x_0} + \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} + \frac{\partial \phi}{\partial x_3}\right) - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \left\{\left(\frac{\partial \phi}{\partial x^1}\right)^2 + \left(\frac{\partial \phi}{\partial x^2}\right)^2 + \left(\frac{\partial \phi}{\partial x^3}\right)^2\right\} - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{c^2}\dot{\phi}^2 - |\vec{\nabla}\phi|^2 - \mu^2\phi^2\right] \to (2)$$

Euler - Lagrange eqn of motion is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right\} = 0$$

Deriving Equation of Motion with Euler-Lagrange's method

if Lagrangian density is given as,

$$\mathcal{L} = \frac{1}{2}[\phi_{,\mu}\phi^{,\mu}_{,} - \mu^2\phi^2)$$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi)(\partial^\mu \phi) - \mu^2\phi^2\right] \to (1)$$

$\because \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ & $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$

$$= \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x^\mu}\right)\left(\frac{\partial \phi}{\partial x_\mu}\right) - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x^0} + \frac{\partial \phi}{\partial x^1} + \frac{\partial \phi}{\partial x^2} + \frac{\partial \phi}{\partial x^3}\right)\right.$$

$$\left.\left(\frac{\partial \phi}{\partial x_0} + \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} + \frac{\partial \phi}{\partial x_3}\right) - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \left\{\left(\frac{\partial \phi}{\partial x^1}\right)^2 + \left(\frac{\partial \phi}{\partial x^2}\right)^2 + \left(\frac{\partial \phi}{\partial x^3}\right)^2\right\}\right.$$

$$\left. - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{c^2}\dot{\phi}^2 - |\vec{\nabla}\phi|^2 - \mu^2\phi^2\right] \to (2)$$

Euler - Lagrange eqn of motion is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right\} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right\} = 0$$

Deriving Equation of Motion with Euler-Lagrange's method contin..

from eqⁿ (1)

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\mu^2 \phi \quad \&$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$\therefore$ from eqⁿ of motion

$$-\mu^2\phi - \partial_\mu [\partial^\mu \phi] = 0$$

$$\rightarrow \quad \partial_\mu \partial^\mu \phi + \mu^2 \phi = 0$$

or $(\partial_\mu \partial^\mu + \mu^2)\phi = 0$

or $\boxed{(\Box + \mu^2)\phi = 0}$

$\because \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box$

To calculate momentum canonical ~~to~~ ϕ or conjugate momentum, we use eqⁿ (2),

$$\because \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \frac{1}{c^2}\dot{\phi}(x)$$

& that Hamiltonian density is

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}$$

$$\Rightarrow \quad \mathcal{H} = \pi(x) \cdot c^2 \pi(x) - \frac{1}{2}\left[\frac{1}{c^2}(c^4 \pi^2(x)) - |\vec{\nabla}\phi|^2 - \mu^2\phi^2\right]$$

$$\mathcal{H} = \frac{1}{2}\left[\pi^2(x)c^2 + |\vec{\nabla}\phi|^2 + \mu^2\phi^2\right] \rightarrow (3)$$

Quantized Lagrangian Field Theory: Canonical Commutation Relations

Quantized Lagrangian field theory: Quantization of K.G field: Canonical commutation relations

Now we can easily move from classical field theory to Quantum field theory by interpreting the conjugate coordinates & conjugate momenta of the discrete lattice approximation given in eqn (8) & (10).

i.e. if we treat (8) & (10) as a Heisenberg operators & subjecting them to the usual canonical commutation relations:

$$\left.\begin{aligned} [\phi_r(j,t), \pi_s(j',t)] &= i\hbar \frac{\delta_{rs}\delta_{jj'}}{\delta x_j} \\ [\phi_r(j,t), \phi_s(j',t)] &= [\pi_r(j,t), \pi_s(j',t)] = 0 \end{aligned}\right\} \quad (21)$$

& also, if we let the lattice spacing go to zero

Quantized Lagrangian Field Theory: Canonical Commutation Relations

then, eqn (21) becomes commutation relations for the fields as:

$$\left.\begin{aligned}[\phi_r(\bar{x},t), \pi_s(\bar{x}',t)] &= i\hbar\,\delta_{rs}\,\delta(\bar{x}-\bar{x}') \\ [\phi_r(\bar{x},t), \phi_s(\bar{x}',t)] &= [\pi_r(\bar{x},t), \pi_s(\bar{x}',t)] = 0\end{aligned}\right\} (22)$$

In the limit of

$\delta \bar{x}_j \to 0$,

$\frac{\delta_{jj}}{\delta x_j} \xrightarrow{\text{becomes}}$ the 3D^l. Dirac delta function $\delta(\bar{x}-\bar{x}')$, in which point $\bar{x}$ & $\bar{x}'$ lying in the jth & j'th cell respect.

Remember that, eqn (22) is canonical commutation relns of the fields at the same time, they are also known as equal - time commutation relns. we can also obtain commutators of the fields at different times.]

for the K - G fields, the eqn (22) can be reduce in to following commutation relations, which is known as canonical commutation relations for K. G. field:

Quantized Lagrangian Field Theory: Canonical Commutation Relations

$$\left.\begin{aligned}
[\phi(\bar{x},t), \dot{\phi}(\bar{x}',t)] &= i\hbar c^2 \delta(\bar{x}-\bar{x}') \\
[\phi(\bar{x},t), \phi(\bar{x}',t)] &= [\dot{\phi}(\bar{x},t), \dot{\phi}(\bar{x}',t)] = 0
\end{aligned}\right\} \quad (23)$$

The equation of motion of this field is the Klein-Gordon equation

$$(\Box + \mu^2)\,\phi(x) = 0 \qquad \longrightarrow (18)$$

The conjugate field is -

$$\pi(x) = \frac{1}{c^2}\,\dot{\phi}(x) \qquad \longrightarrow (19)$$

and the Hamiltonian density is

$$\mathcal{H}(x) = \frac{1}{2}\left[c^2\pi^2(x) + (\nabla\phi)^2 + \mu^2\phi^2\right] \qquad \longrightarrow (20).$$

<u>Quantized Lagrangian Field Theory</u>:-

Now it is easy to go from the classical to the quantum field theory by interpreting the conjugate coordinates and momenta of the the discrete lattice approximation, equation (8) and (10), as Heisenberg operators, and subjecting these to the usual canonical commutation relations:

$$\left.\begin{aligned} [\phi_r(j,t), \pi_s(j',t)] &= i\hbar \frac{\delta_{rs}\,\delta_{jj'}}{\delta x_j} \\ [\phi_r(j,t), \phi_s(j',t)] &= [\pi_r(j,t), \pi_s(j',t)] = 0 \end{aligned}\right\} \rightarrow (21)$$

let the lattice spacing go to zero, then (21) becomes -

$$\left.\begin{aligned} [\phi_r(\bar{x},t), \pi_s(\bar{x}',t)] &= i\hbar\,\delta_{rs}(\bar{x}-\bar{x}') \\ [\phi_r(\bar{x},t), \phi_s(\bar{x}',t)] &= [\pi_r(\bar{x},t), \pi_s(\bar{x}',t)] = 0 \end{aligned}\right\} \rightarrow (22)$$

In the limit, as $\delta \bar{x}_j \to 0$, $\frac{\delta jj'}{\delta \bar{x}_j}$ becomes the three-dimensional Dirac delta function $\delta(\bar{x}-\bar{x}')$, the points $\bar{x}$ and $\bar{x}'$ lying in the j^{th} and j'^{th} cell respectively.

For the Klein-Gordan fields, equation (22) reduce to the commutation relations—

$$[\phi(\bar{x},t), \dot{\phi}(\bar{x}', t)] = i\hbar c^2 \delta(\bar{x}-\bar{x}')$$

$$[\phi(\bar{x},t), \phi(\bar{x}',t)] = [\dot{\phi}(\bar{x},t), \dot{\phi}(\bar{x}',t)] = 0 \longrightarrow (23)$$

THE KLEIN-GORDAN FIELD.

: The Real Klein-Gordan Field :— (for spin-0)
For particle of rest mass m, energy momentum are related by—

$$E^2 = m^2c^4 + c^2\bar{p}^2 \longrightarrow (1)$$

In quantum mechanics—

$$\bar{p} \longrightarrow -i\hbar \bar{\nabla}, \ \bar{E} \to i\hbar \partial/\partial t$$

Then we have,

$$(\Box + \mu^2)\phi(x) = 0 \longrightarrow (2)$$

where $\mu \equiv mc/\hbar$

From equation (1), energy eigen values are—

$$E^2 = p^2 + m^2 \Rightarrow E = \pm\sqrt{p^2+m^2}$$

(In the unit $c = 1$)

Normal Ordering of Operators:

Normal ordering of operators:-

In normal ordering, we encounter ~~witth~~ with product of creation and annihilation operators.

Therefore, in quantum field theory a product of quantum fields (or equivalently their creation & annihilation operators) is said to be normal ordered (or wick order) when all creation operators are placed ~~to the~~ left ~~of~~ to the all annihilation operators in the product.

or

we define a normal-ordered product by moving all annihilation operators to the right of all creation operators.

Hence, the process of putting a product in to normal order is called normal ordering (or wick ordering).

Normal Ordering of Operators continu…

The term antinormal ordering are also analogously defined, where the annihilation operators are placed to the left of the creation operators.

Importance :– The process of normal ordering is particularly important for a quantum mechanical Hamiltonian. When quantizing a classical Hamiltonian there is some freedom when choosing the operator order, and these choices lead to differences in the ground state energy.

Notation :→ The normal product is denoted by $N(\hat{O})$ or $N(\cdots)$ where $\hat{O}$ denotes → arbitrary product of creation &/or annihilation operators (or equivalently, quantum fields).

. The normal ordered form of $\hat{O}$ is also denoted as $:\hat{O}:$.

Note that :→ The concept of normal ordering makes sense only for products of operators. The attempt to apply normal

Normal Ordering of Operators continu…

ordering [illegible] a sum of operators is not useful.

Example :- (i)

$$\boxed{\begin{aligned} N\left(a(\bar{k}_1)\,a(\bar{k}_2)\,a^{\dagger}(\bar{k}_3)\right) &= a^{\dagger}(\bar{k}_3)\,a(\bar{k}_1)\,a(\bar{k}_2) \\ \text{or} \quad &= a^{\dagger}(\bar{k}_3)\,a(\bar{k}_2)\,a(\bar{k}_1) \end{aligned}} \quad (18)$$

{∵ Normal ordering does not fix the order of annihilation or creation operators hence both above are equal }.

(ii)

$N[\phi(x)\,\phi(y)]$.

$$= N\left[\left(\phi^{+}(x) + \phi^{-}(x)\right)\left(\phi^{+}(y) + \phi^{-}(y)\right)\right]$$

$$= N[\phi^{+}(x)\,\phi^{+}(y)] + N[\phi^{+}(x)\,\phi^{-}(y)] + N[\phi^{-}(x)\,\phi^{+}(y)] + N[\phi^{-}(x)\,\phi^{-}(y)]$$

$$= \phi^{+}(x)\,\phi^{+}(y) + \phi^{-}(y)\,\phi^{+}(x) + \phi^{-}(x)\,\phi^{+}(y) + \phi^{-}(x)\,\phi^{-}(y) \quad (19)$$

In above, the order of the factors has been interchanged in second term only, because ϕ^{+} → which is the +ve frequency parts, contain only annihilation operators & this should stand to the right of all -ve freq. parts ϕ^{-} → which contain only

Normal Ordering of Operators continu…

creation operators.

Use in quantum field theory :-

(i) ~~The~~ To calculate the vacuum expectation value of a normal ordered product of creation & annihilation operators :

The vacuum expectation value of any normal ordered products or of creatⁿ & annihilation operators is zero.

Because :-

if we denote $|0\rangle$ as vacuum state, then ~~creation~~ & annihilation & creation operators satisfy ;

$$\boxed{\begin{aligned} \hat{a}\,\langle 0| &= 0 \\ \langle 0|\,\hat{a}^{\dagger} &= 0 \end{aligned}}$$ &

when $\hat{a} \longrightarrow$ annihilation operator
& $\hat{a}^{+} \longrightarrow$ creation operators
{ either bosonic or fermionic

The examples of particles of Klein-Gordon field are bosons.

Fock State:-

Fock state :- In q.m.,
a Fock state or number state is
a quantum state that is an
element of a fock space with
a well defined number of particles
(or quanta) [named after physicist
Vladimir fock].
Fock state play an important
role in the second quantization
formulation of quantum mechanics.

useful to distinguish 2nd quantiz from 1st quantization

Fock Space :

it is used to construct the
quantum states space of a
variable or unknown no. of
identical particles from a
single particle Hilbert space
H.

Informally, a Fock space is the
sum of a set of Hilbert spaces
representing zero particle states,
one particle states, two particle
states, & so on.

A general state in fock
space is a linear combination
of n-particle states, one for
each n.

Fock State Continue..:

As we know that, the vacuum state is a state with no particles. But when we start to describe any physical events, we need other states, in particular states with specified particle content.

To defined such kind of states we can use analogy of excited states of oscillators.

e.g.,

One can define a one-particle state as

$$|p\rangle \equiv a^{\dagger}(p)|0\rangle \longrightarrow (1)$$

The above state contains one quantum of the field ϕ with momentum $p^{\mu} = (E_p, p)$.

& such states have positive norms since $\langle p|p'\rangle = \delta^3(p-p') \rightarrow (2)$

which comes from the commutation relation & the definition of vacuum.

Similarly, we can define many particle states.

if a state has 'N' particles with all different momenta $p_1, p_2, \ldots, p_N$

Fock State Continue..:

it can be defined by

$$|p_1, p_2, \ldots p_N\rangle = a^\dagger(p_1)\, a^\dagger(p_2) \ldots a^\dagger(p_N)|0\rangle \quad (3)$$

Now, if we want to construct a state with n-particle of momentum p, it will be given by followings:-

$$\boxed{|p(n)\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger(p)\right)^n |0\rangle} \quad (4)$$

where the prefactor is needed for proper normalization.

Such kind of multi-particle states distinguish second quantization (also kn- as field quantifying) from first quantization (also kn- as single-particle quantum mechanics.).

The vacuum, together with single particle states and all multi-particle states, constitute a vector space which is called as fock space.

The creation & annihilation operators act on this space.