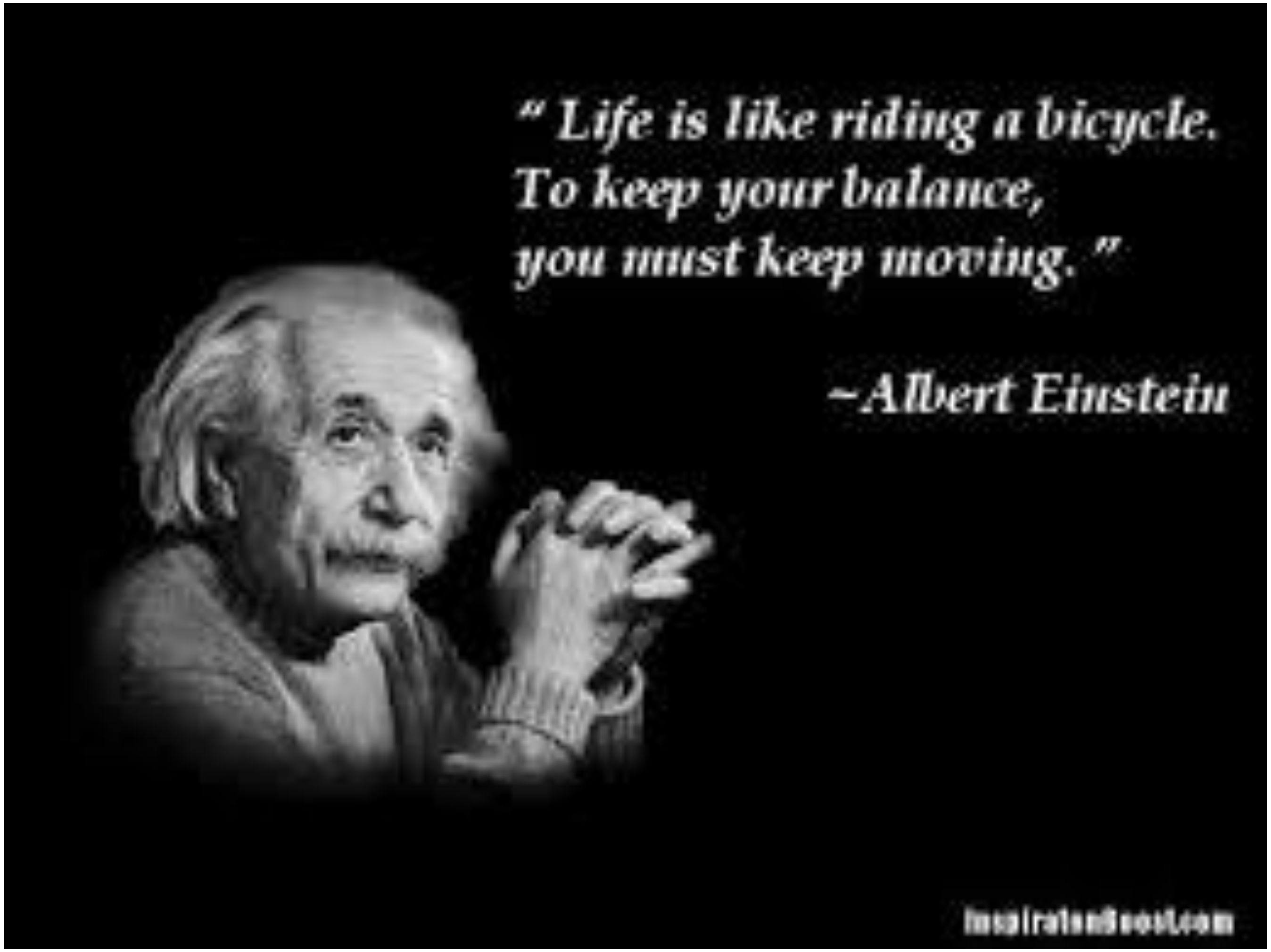
Theory of Relativity

UNIT I
Relativistic Mechanics

Lecture-7


" Life is like riding a bicycle.
To keep your balance,
you must keep moving. "
-Albert Einstein

MASS–ENERGY EQUIVALENCE

Solution

From the expression of the relativistic mass of the particle, we know that mass,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{m_0}{\sqrt{1 - \frac{c^2}{2c^2}}} = \sqrt{2}\, m_0 = 1.41 m_0$$

The momentum P of the particle is given by

$$P = mv = m_0 \sqrt{2} \times \frac{c}{\sqrt{2}} = m_0 c$$

The total energy E of the particle is given by

$$E = mc^2 = (1.41 m_0)\, c^2 = 1.41 m_0 c^2$$

The kinetic energy K of the particle is given by

$$K = E - m_0 c^2 = 1.41 m_0 c^2 - m_0 c^2$$

$$= 0.41 m_0 c^2$$

Solution

The kinetic energy of the particle is given by

or $$\upsilon = c\sqrt{1 - \left(\frac{m_0}{m_0}\right)^2}$$

$$= 3 \times 10^8 \sqrt{1 - \left(\frac{m_0}{11 m_0}\right)^2}$$

$$= 3 \times 10^8 \times 0.996$$

$$= 2.99 \times 10^8 \text{ m/s}$$

Hence, $$P = 11\, m_0 \times 2.99 \times 10^8 = 11 \times 9.11 \times 10^{-31} \times 2.99 \times 10^8$$

$$= 2.99 \times 10^{-21} \text{ kg m/s}$$

$$m = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} \quad \text{or} \quad 1 - \frac{\upsilon^2}{c^2} = \left(\frac{m_0}{m}\right)$$

- Example-3
- How much does a proton gain in mass when accelerated to a kinetic energy of 500 MeV?

Solution

Kinetic energy $K = (m - m_0)\, c^2$

or $mc^2 = K + m_0 c^2$

Gain in mass $m - m_0 = \dfrac{K}{c^2} = \Delta m$

Here, it is given that

$$K = 500 \text{ MeV}$$

$$= 500 \times 10^6 \times 1.6 \times 10^{-19} \text{ J}$$

$$m_0 = 1.6 \times 10^{-27} \text{ kg} \quad \text{and} \quad c = 3 \times 10^8 \text{ m/s}$$

$$\Delta m = \frac{500 \times 10^6 \times 1.6 \times 10^{-19}}{(3 \times 10^8)^2} = 8.89 \times 10^{-28} \text{ kg}$$

- Example-4: Find the speed of 0.1 MeV electrons according to the classical and relativistic mechanics.

Solution

According to the classical mechanics, the kinetic energy is expressed as

$$K = \frac{1}{2} m\upsilon^2$$

or $$\upsilon = \sqrt{2K/m}$$

Given that $K = 0.1$ MeV $= 0.1 \times 10^6$ eV $= 0.1 \times 10^6 \times 1.6 \times 10^{-19}$ J and $m = 9.1 \times 10^{-31}$ kg.

Now, $$\upsilon = \sqrt{\frac{2 \times 0.1 \times 10^6 \times 1.6 \times 10^{-19}}{9.11 \times 10^{-31}}}$$

$$= 1.87 \times 10^8 \text{ m/s}$$

According to the relativistic mechanics, the KE of an electron is expressed as

$$K = mc^2 - m_0c^2$$

or $$mc^2 = K + m_0c^2$$

or $$\frac{m_0c^2}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} = K + m_0c^2$$

or $$\left(1 - \frac{\upsilon^2}{c^2}\right) = \left[\frac{m_0c^2}{K + m_0c^2}\right]^2$$

or $$\upsilon = c\sqrt{1 - \left(\frac{m_0c^2}{K + m_0c^2}\right)^2}$$

The rest-mass energy of electron is m_0c^2, i.e., 0.512 MeV.

and $$K + m_0c^2 = 0.1 \text{ MeV} + 0.512 \text{ MeV} = 0.612 \text{ MeV}$$

Hence, $$\upsilon = 3.0 \times 10^8 \sqrt{1 - \left(\frac{0.512}{0.612}\right)^2} = 3 \times 10^8 \times 0.548$$

$$= 1.64 \times 10^8 \text{ m/s}$$

Show that the momentum of a particle of rest mass m_0 and kinetic energy KE is given by the expression

$$p = \sqrt{\frac{\text{KE}^2}{c^2} + 2m_0 \times \text{KE}}\ .$$

Solution

From the energy–momentum relation, we know that

$$E^2 = m_0{}^2c^4 + p^2c^2$$

or
$$E = \left(m_0^2c^4 + p^2c^2\right)^{1/2}$$

The total energy E = rest-mass energy + kinetic energy

$$E = m_0c^2 + \text{KE}$$

From Eqs. (1) and (2),

$$\left(m_0^2c^4 + p^2c^2\right)^{1/2} = m_0c^2 + \text{KE}$$

Squaring both sides of this equation, we get

$$m_0{}^2c^4 + p^2c^2 = m_0{}^2c^4 + 2m_0c^2 \times \text{KE} + \text{KE}^2$$

or
$$p^2c^2 = \text{KE}^2 + 2m_0c^2 \times \text{KE}$$

or
$$p^2 = \frac{\text{KE}^2}{c^2} + 2m_0 \times \text{KE}$$

or
$$p = \sqrt{\frac{\text{KE}^2}{c^2} + 2m_0 \times \text{KE}}$$

Example-5

Show that the massless particles can exist only if they move with the speed of light and their energy E and momentum P must have the relation E = Pc.

Solution

A particle which has zero rest mass (m_0) is called a massless particle. In classical physics, such particles do not exist, while in relativistic mechanics, such particles may exist.

From the relation of relativistic energy and momentum, we know that

$$E = \sqrt{m_0^2 c^4 + P^2 c^2}$$

For massless particles, $m_0 = 0$

Thus, $E = Pc$

or $P = \frac{E}{c}$

Since $P = m\upsilon$

so for massles particles,

$$P = \frac{E}{c}$$

$$= \frac{mc^2}{c}$$

$$= mc$$

It shows that the massless particles have the same velocity as light in free space.

Example-6: Show that for small velocities, the relativistic kinetic energy of a body reduces to the classical kinetic energy, which is less than the rest-mass energy.

Solution

From the expression of relativistic kinetic energy, we know that

$$KE_{\text{relativistic}} = (m - m_0)c^2$$

$$= \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} - m_0 c^2$$

$$= m_0 c^2 \left[\left(1 - \frac{\upsilon^2}{c^2} \right)^{-1/2} - 1 \right]$$

$$= m_0 c^2 \left[1 + \frac{\upsilon^2}{2c^2} + \cdots - 1 \right]$$

Now, neglecting higher-order terms because $\upsilon << c$, we get

$$KE_{\text{relativistic}} = m_0 c^2 \times \frac{\upsilon^2}{2c^2}$$

$$= \frac{1}{2} m_0 \upsilon^2$$

$$= KE_{\text{classical}}$$

which is less than $m_0 c^2$.

Show that the velocity at which the mass of a particle is increased to n times its rest mass is $\left(\sqrt{n^2-1}/n\right)c$.

For what value of v/c $(=\beta)$ will the relativistic mass of a particle exceed its rest mass by a given ratio R.

[*Hint*: $R=(m-m_0)/m_0$]

Assignment

- How fast must an electron move in order that its mass equals the rest mass of the proton.
- If the kinetic energy of a body is thrice of its rest-mass energy, then find its velocity.
- If the total energy of a particle is exactly thrice of its rest-mass of energy, what is the velocity of the particle?
- If the kinetic energy of a body is thrice of its rest-mass energy, then find its velocity.
- Calculate the amount of work done to increase the speed of an electron from 0.6c to 0.8c. Given that the rest-mass energy of electron = 0.511 MeV.

Show that if the variation of mass with velocity is taken into account, the kinetic energy of a particle of rest mass m_0 and moving with velocity υ is given by

$$\mathrm{KE} = m_0 c^2 \left[\left(1 - \frac{\upsilon^2}{c^2} \right)^{-1/2} - 1 \right]$$