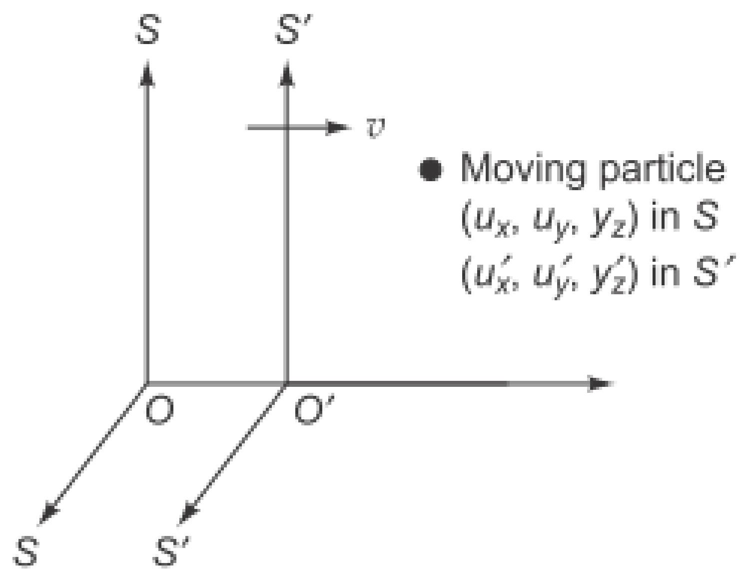
Theory of Relativity

UNIT I
Relativistic Mechanics
Lecture-5

ऑधियो की जद मे रहकर तो देखो
फनूस भी तीर नजर आयेगे
रोशनी की तर्ज पर चलकर तो देखो
खाक के जर्रे कहकशा बन जायेंगे

Transformation of Velocity Components—Velocity Addition

- The transformation equations relating the components of velocity observed by the observer in the moving and stationary frames of reference are known as *addition of velocities.*



Transformation of Velocity Components—Velocity Addition Contd….

- $u_x = \frac{dx}{dt}$ $\quad u_y = \frac{dy}{dt}$ $\quad u_z = \frac{dz}{dt}$
- $u'_x = \frac{dx'}{dt'}$ $\quad u'_y = \frac{dy'}{dt'}$ $\quad u'_z = \frac{dz'}{dt'}$

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \; y = y', \; z = z', \text{ and } t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Transformation of Velocity Components—Velocity Addition Contd….

$$dx = \gamma(dx' + \upsilon dt'), \quad dy = dy', \quad dz = dz', \quad \text{and} \quad dt = \gamma\left(dt' + \frac{\upsilon}{c^2}dx'\right)$$

Now,
$$\frac{dx}{dt} = \frac{dx' + \upsilon dt'}{dt' + \frac{\upsilon dx'}{c^2}}$$

$$= \frac{dt'\left(\frac{dx'}{dt'} + \upsilon\right)}{dt'\left(1 + \frac{\upsilon}{c^2}\frac{dx'}{dt'}\right)}$$

or
$$\frac{dx}{dt} = \frac{\frac{dx'}{dt'} + \upsilon}{1 + \frac{\upsilon}{c^2}\frac{dx'}{dt'}}$$

Transformation of Velocity Components—Velocity Addition Contd….

$$u_x = \frac{u'_x + \upsilon}{1 + \frac{\upsilon}{c^2} u'_x}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{\upsilon}{c^2} dx'\right)}$$

$$= \frac{u'_y}{\gamma\left(1 + \frac{\upsilon}{c^2} u'_x\right)}$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + \frac{\upsilon}{c^2} dx'\right)}$$

$$= \frac{dz'}{\gamma\, dt'\left(1 + \frac{\upsilon}{c^2} \frac{dx'}{dt'}\right)}$$

$$= \frac{u'_z}{\gamma\left(1 + \frac{\upsilon}{c^2} u'_x\right)}$$

Transformation of Velocity Components— Velocity Addition Contd.…

- Alternative Method

Differentiating the expression of t with respect to t', we get

$$\frac{dt}{dt'} = \frac{1+\frac{\upsilon}{c^2}\frac{dx'}{dt'}}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

or
$$\frac{dt'}{dt} = \gamma\left(1+\frac{\upsilon u'_x}{c^2}\right)$$

where
$$\gamma = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

We know that

$$u_x = \frac{d(x)}{dt}$$

$$= \frac{d}{dt}\left[\gamma\left(x'+\upsilon t'\right)\right]$$

$$= \gamma\left[\frac{dx'}{dt'}\cdot\frac{dt'}{dt}+\frac{\upsilon dt'}{dt}\right]$$

$$= \gamma\left[u_x'+\upsilon\right]\frac{dt'}{dt}$$

Alternative Method Contd…..

$$u_x = \frac{\gamma(u_x' + v)}{\gamma\left(1 + \frac{v u_x'}{c^2}\right)}$$

$$u_x = \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}}$$

Similarly $$u_z = \frac{u_z'}{\gamma\left(1 + \frac{v u_x'}{c^2}\right)}$$

Now, $$u_y = \frac{d(y)}{dt} = \frac{dy'}{dt'}\frac{dt'}{dt}$$

Again putting the value of dt'/dt in the above equation, we get

$$u_y = \frac{u_y'}{\gamma\left(1 + \frac{v u_x'}{c^2}\right)}$$

$$u_x' = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}, \quad u_y' = \frac{u_y}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}, \quad u_z' = \frac{u_z}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}$$

Verification of constancy of speed of light

- Let us consider a particle (photon) moving with velocity c in frame S', i.e., $u_x' = c$. Let us put this value of u_x' in following equation to obtain the value of u_x

$$u_x = \frac{u_x' + v}{1 + \frac{v}{c^2} u_x'}$$

$$u_x = \frac{c + v}{1 + \frac{vc}{c^2}}$$

$$u_x = \frac{(c + v)}{\frac{(c + v)}{c}} = c$$

VARIATION OF MASS WITH VELOCITY

- According to the Newtonian mechanics, the mass (inertia) of a body is constant and it is independent of its velocity.

- In the special theory of relativity, it is observed that the law of velocity addition (components of velocity) is not same as it is in the Newtonian or Galilean theory of relativity.

- Since momentum is the product of the mass and the velocity, so the law of conservation of momentum will be violated if the special theory of relativity is taken into account.

VARIATION OF MASS WITH VELOCITY CONTD….

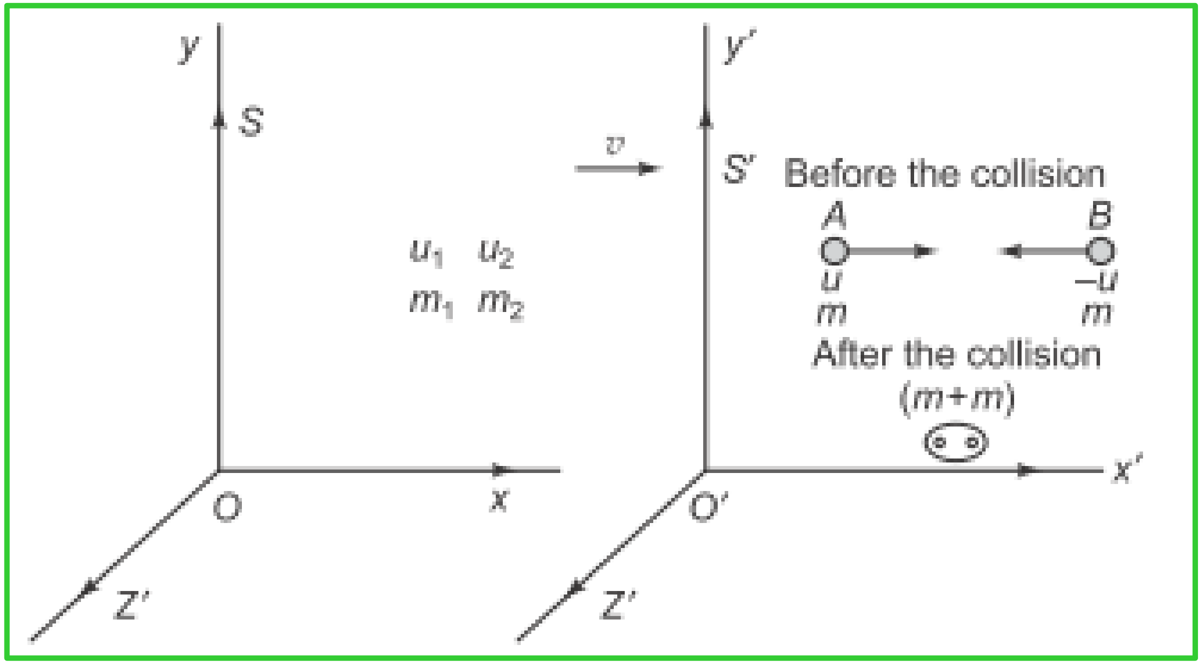
- If anyone wants to preserve the law of conservation of momentum, then it can be done only at the cost of variation of mass with velocity according to the relation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- In order to derive the expression for the variation of mass, let us consider two inertial frames of reference S and S', where S' is moving with constant velocity u relative to S in the positive x-direction.

VARIATION OF MASS WITH VELOCITY CONTD….

- Suppose there are two elastic balls A and B exactly similar having same mass m in frame S'. These balls are approaching each other with the same speed (i.e., the ball A has its velocity u while the ball B has its velocity –u). When the observer of frame S (i.e., O) records the mass and velocity of balls A and B, he finds that it is m_1, u_1 and m_2, u_2 for the balls A and B, respectively,



VARIATION OF MASS WITH VELOCITY CONTD….

- When we apply the law of conservation of momentum for balls A and B in S' frame, then we can write

- momentum of ball A + momentum of ball B = momentum of coalesced mass,

- mu + (–mu) = (m + m) x 0

- Now, when we apply the law of conservation of momentum for balls A and B by being in S frame of reference, then we get

$$m_1\, u_1 + m_2\, u_2 = (m_1 + m_2)v$$

- Using the values of u_1 and u_2

VARIATION OF MASS WITH VELOCITY CONTD….

$$u_1 = \frac{u+v}{1+\frac{uv}{c^2}} \quad \text{and} \quad u_2 = \frac{-u+v}{1-\frac{uv}{c^2}}$$

Using these values of u_1 and u_2 in Eq. (1.32), we get

$$m_1\left(\frac{u+v}{1+\frac{uv}{c^2}}\right) + m_2\left(\frac{-u+v}{1-\frac{vu}{c^2}}\right) = (m_1+m_2)\,v$$

VARIATION OF MASS WITH VELOCITY CONTD….

$$m_1\left[\frac{u+v}{1+\frac{uv}{c^2}}-v\right]=m_2\left[v-\frac{-u+v}{1-\frac{uv}{c^2}}\right]$$

$$m_1\left[\frac{u+v-v-\frac{uv^2}{c^2}}{1+\frac{uv}{c^2}}\right]=m_2\left[\frac{v-\frac{uv^2}{c^2}+u-v}{1-\frac{uv}{c^2}}\right]$$

$$m_1\left[\frac{u\left(1-\frac{v^2}{c^2}\right)}{1+\frac{uv}{c^2}}\right]=m_2\left[\frac{u\left(1-\frac{v^2}{c^2}\right)}{1-\frac{uv}{c^2}}\right]$$

$$\frac{m_1}{m_2}=\frac{1+\frac{uv}{c^2}}{1-\frac{uv}{c^2}}$$

VARIATION OF MASS WITH VELOCITY CONTD….

$$1-\frac{u_1^2}{c^2}=1-\frac{\left(\frac{u+v}{c}\right)^2}{\left(1+\frac{uv}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2}=\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{uv}{c^2}\right)^2}$$

$$1-\frac{u_2^2}{c^2}=\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{uv}{c^2}\right)^2}$$

$$\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}=\frac{\left(1+\frac{uv}{c^2}\right)^2}{\left(1-\frac{uv}{c^2}\right)^2}$$

$$\sqrt{\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}}=\frac{1+\frac{uv}{c^2}}{1-\frac{uv}{c^2}}$$

$$\frac{m_1}{m_2}=\frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

$$m_1\sqrt{1-\frac{u_1^2}{c^2}}=m_2\sqrt{1-\frac{u_2^2}{c^2}}$$

VARIATION OF MASS WITH VELOCITY CONTD….

Since the LHS and RHS of the above equation are independent of each other, so both can be equated to a constant quantity, i.e.,

$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} = m_0$$

where m_0 is the rest mass of ball A.

Thus,
$$m_1 = \frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

and
$$m_2 = \frac{m_0}{\sqrt{1-\frac{u_2^2}{c^2}}}$$

Hence, for a body having its rest mass m_0 moving with velocity v, we can write

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.38}$$

where m is the mass of the body during its motion.

Mass:

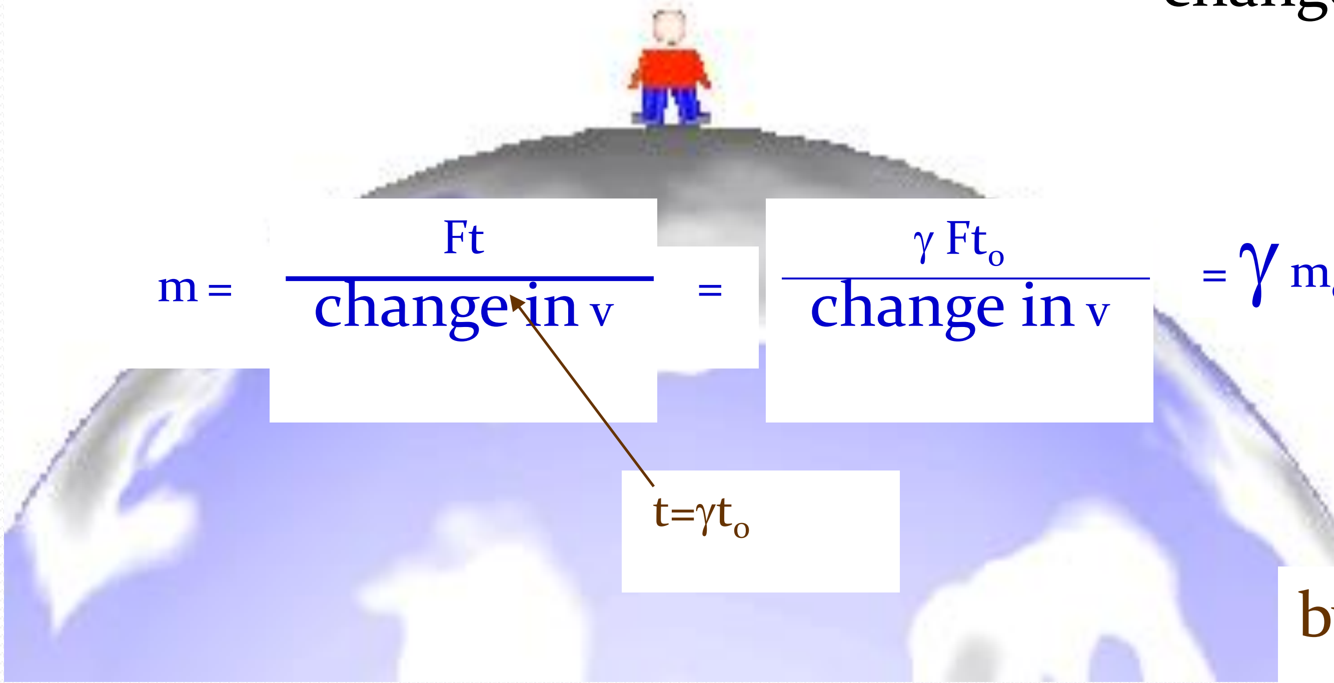
$F = m_o a \quad = m_o \dfrac{\text{change in } v}{\text{time}}$

a

time $= t_o$ → t_o

$$\text{change in } v = \frac{F t_o}{m_o}$$

$$m_o = \frac{F t_o}{\text{change in } v}$$



$$m = \frac{F t}{\text{change in } v} = \frac{\gamma F t_o}{\text{change in } v} = \gamma m_o$$

$t = \gamma t_o$

mass increases!!

$$m = \gamma m_o$$

by a factor γ

Relativistic mass increase

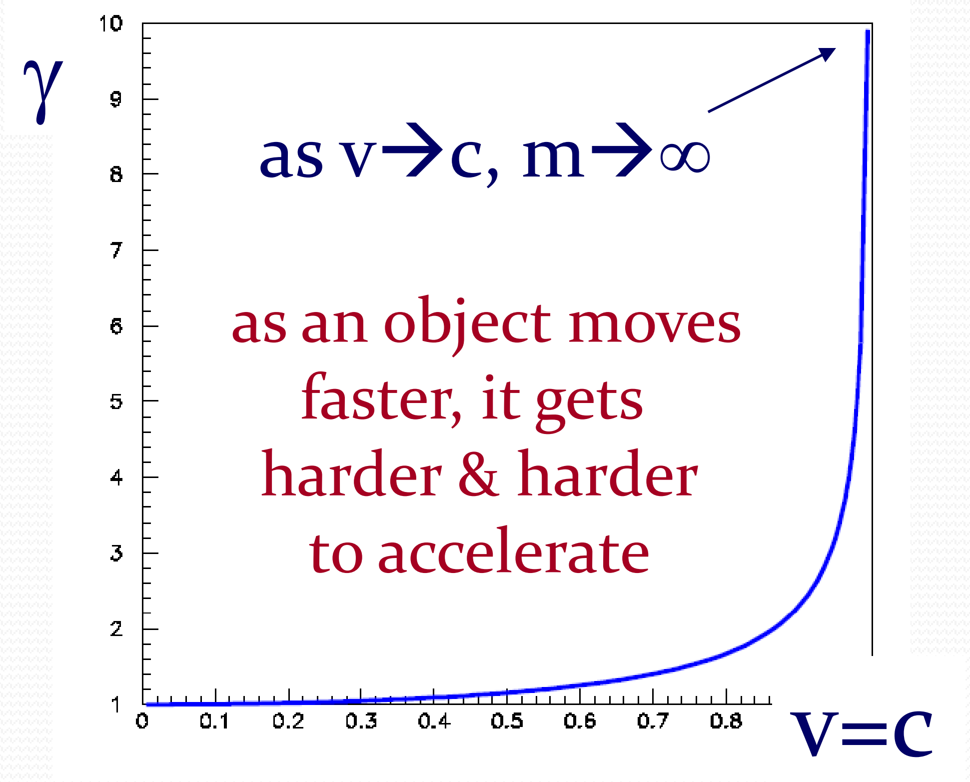
m_o = mass of an object when it is at rest

➔ "rest mass"

mass of a moving object increases

$$m = \gamma\, m_o$$

by the γ factor



Conceptual questions

Show that if in the S' frame we have $u_y' = c \sin\phi$ and $u_x' = c\cos\phi$, then in frame S, $u_x^2 + u_y^2 = c^2$, where S' is moving with velocity υ relative to S.

Solution

Frame S' is moving with velocity υ relative to S in +ve x-direction. Using the inverse equation of addition of velocity, we can write

$$u_x = \frac{u_x' + \upsilon}{1 + \frac{u_x' \upsilon}{c^2}} \quad \text{and} \quad u_y = \frac{u_y' \sqrt{1 - \frac{\upsilon^2}{c^2}}}{1 + \frac{u_x' \upsilon}{c^2}}$$

Now, putting $u_x' = c\cos\phi$ and $u'_y = c\sin\phi$, we have

$$u_x^2 + u_y^2 = \left[\frac{c\cos\phi + \upsilon}{1 + \frac{\upsilon\cos\phi}{c}}\right]^2 + \frac{c^2\sin^2\phi\left(1 - \frac{\upsilon^2}{c^2}\right)}{\left(1 + \frac{\upsilon\cos\phi}{c}\right)^2}$$

$$= \frac{1}{\left(1 + \frac{\upsilon\cos\phi}{c}\right)^2}\left[c^2 + 2c\upsilon\cos\phi + \upsilon^2(1 - \sin^2\phi)\right]$$

$$= \frac{1}{\left(1 + \frac{\upsilon\cos\phi}{c}\right)^2}\left[c^2 + 2c\upsilon\cos\phi + \upsilon^2\cos^2\phi\right]$$

$$= \frac{c^2}{\left(1 + \frac{\upsilon\cos\phi}{c}\right)^2}\left[1 + \frac{2\upsilon\cos\phi}{c} + \frac{\upsilon^2}{c^2}\cos^2\phi\right]$$

$$= \frac{c^2}{\left(1 + \frac{\upsilon\cos\phi}{c}\right)^2}\left[1 + \frac{\upsilon\cos\phi}{c}\right]^2$$

$$= c^2$$

Conceptual Questions

What is the length of a metre stick moving parallel to its length when its mass is 3/2 times its rest mass.

Solution

Given that

$$m = \frac{3}{2}\ m_0$$

Initial length of the metre stick is 1 m. Hence, the contracted length of the metre stick can be given as

$$l = l_0\sqrt{1 - \frac{v^2}{c^2}}$$

$$= l_0\sqrt{1 - \frac{5c^2}{9c^2}}$$

$$= l_0\sqrt{\frac{4}{9}}$$

$$= \frac{2l_0}{3}$$

For $l_0 = 1$ m

$$l = \frac{2}{3}\ \text{m}$$

$$= 0.667\ \text{m}$$

Using the relativistic formula of mass, we can write

$$\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{3}{2}m_0$$

$$1 - \frac{v^2}{c^2} = \frac{4}{9}$$

$$v = c\sqrt{1 - \frac{4}{9}}$$

$$= c\sqrt{\frac{5}{9}}$$

$$= \frac{c\sqrt{5}}{3}$$

Assignments based on what we learnt in this lecture

- How the velocity component changes with velocity?
- How these are different from the Galilean transformation of velocity component?
- Show that the addition of velocity component prove the constancy of speed of light.
- How the relativistic velocity addition theorem suggest the variation of mass with velocity.
- Using the concept of theory of relativity obtain the expression of variation of mass with velocity.
- Explain the concept of infinite mass of a body moving with velocity of light.