UNIT-V

Electromagnetism

CHAPTER-16

Maxwell's Equations and Electromagnetic Waves

SCALAR AND VECTOR FIELDS:

The region of space in a point function specifies a physical quantity is known as a field.

- A ***scalar field*** is defined as continuous scalar point function which gives the value of a physical quantity.

- A ***vector field*** is specified by a continuous vector point function having magnitude and direction, both of which change from point to point, in the given region of field.

GRADIENT, DIVERGENCE, AND CURL:

- To study about the rate of change of scalar and vector fields, a common operator called ***del*, or *nabla***, is used, which is written as

$$\vec{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

- If *f* (*x*, *y*, *z*) is a differentiable scalar function, its ***gradient*** is defined as

$$\text{grad}\,\phi = \nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

➢If F is a vector point function (F = $F1i + F2\,j + F3k$), where $F1$, $F2$, and $F3$ are functions
of x, y, and z, then its ***divergence*** written as div F, is given by

$$\vec{\nabla} \cdot \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (F_1 i + F_2 j + F_3 k)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \begin{bmatrix} i \cdot i = j \cdot j = k \cdot k = 1 \\ i \cdot j = j \cdot k = k \cdot i = 0 \end{bmatrix}$$

➢If F is a vector point function ($F = F1\,i + F2\,j + F3k$), where $F1$, $F2$, and $F3$ are functions
of x, y, and z, then its ***curl*** is defined as

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - j\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

- A vector field F is called ***solenoidal*** if divergence F = 0.
- A vector field F is called ***irrotational*** if curl F = 0.

Gauss Divergence Theorem: This theorem states that the flux of a vector field F, over any closed surface *S*, is equal to the volume integral of the divergence of that vector field over the volume *V* enclosed by the surface *S*.

$$\int_S \vec{F} \cdot d\vec{S} = \int_V \text{div}\, \vec{F}\, dV$$

Stokes Theorem: This theorem states that the surface integral of the curl of a vector field *A*, taken over any surface *S*, is equal to the line integral of A around the closed curve forming the periphery of the surface.

$$\iint_S (\text{Curl}\, \vec{A}) \cdot d\vec{S} = \oint \vec{A} \cdot \vec{dl}$$

Poisson's and Laplace's Equations: Poisson's and Laplace's equations are very useful mathematical relations for the calculations of electric fields and potentials that cannot be computed by using Coulomb's and Gauss's law in electrostatic problems.

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$ ………… *Poisson's equation*

$$\nabla^2 V = 0$$ ………… *Laplace's equation*

FUNDAMENTAL LAWS OF ELECTRICITY AND MAGNETISM:

(i) Gauss's law in electrostatics:

$$\oint \vec{E} \cdot \vec{dS} = q/\epsilon_0$$

(ii) Gauss's law in magnetostatics:

$$\oint \vec{B} \cdot \vec{dS} = 0$$

(iii) Faraday's law of electromagnetic induction:

$$e = -\frac{d\phi}{dt}$$

(iv) Ampere's law:

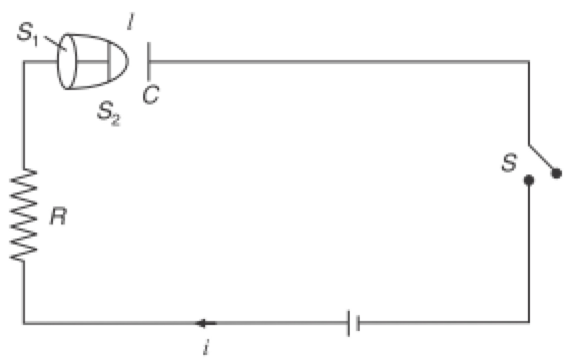
$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

EQUATION OF CONTINUITY: If J is the current density and ρ is the volume charge density then for an arbitrary surface, the expression for continuity equation is given by

$$\int_V \operatorname{div} \vec{J}\, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

DISPLACEMENT CURRENT: ***displacement current*.** A changing electric field in vacuum or in a dielectric also produces a magnetic field. This implies that a changing electric field is equivalent to a current, which flows till the electric field is changing. This equivalent current produces the same magnetic effects as a conventional current in a conductor is known as

Modified Ampere's Law: The concept of displacement current due to the discharge of a condenser leads to the modification in Ampere's law. Modified Ampere's law becomes:



Let us consider a plane surface S_1 and a hemispherical surface S_2 around the condenser plate as shown in Fig. 17.1. Let both surfaces be bounded by the same closed path l, and applying Ampere's law to the surface S_1, we get

$$\oint_{s_1} \vec{B} \cdot d\vec{l} = \mu_0 i$$

and

$$\oint_{s_2} \vec{B} \cdot d\vec{l} = 0$$

Maxwell introduced the idea that a changing electric field is a source of magnetic field in the gap between the capacitor plates (during charging) and is equivalent to the displacement current devalued by $i_{d.}$ If ϕ_E is the electric flux, then from equation of continuity, i_d should be equal to $\epsilon_0 = d\phi_E / dt$.. Therefore, if along with an electric current, there exists a magnetic field, the modified Ampere's law becomes

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(i + \epsilon_0 \frac{d\phi_E}{dt} \right) = \mu_0 (i + i_d)$$

The displacement current in the gap is identical to the conduction current in the connecting wires, we can write,

$$i_d = A \frac{d(\epsilon_0 E)}{dt} = A \frac{dD}{dt}$$

$$\frac{i_d}{A} = \frac{dD}{dt}$$

$$J_d = \frac{dD}{dt}$$

Hence, modified Ampere's law becomes

$$\oint \bar{B} \cdot d\vec{l} = \mu_0 \left(i + \epsilon_0 \frac{d\phi_E}{dt} \right)$$

MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM:

(i) $\vec{\nabla} \cdot \vec{D} = \rho$ or Div $\vec{D} = \rho$

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ or Div $\vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ or Curl $\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ or Curl $\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

MAXWELL'S EQUATIONS IN INTEGRAL FORM:

(i) $\int_s \vec{D} \cdot \vec{dS} = \int_V \rho dV$ or $\oint_s \vec{E} \cdot \vec{dS} = q$

(ii) $\oint_s \vec{B} \cdot \vec{dS} = 0$

(iii) $\oint \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \int \vec{B} \cdot \vec{dS}$

(iv) $\oint \vec{H} \cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{dS}$

PHYSICAL SIGNIFICANCE OF MAXWELL'S EQUATIONS:

(i) ***Maxwell's first equation*** represents the Gauss' law in electrostatics for the static charges, which states that the electric flux through any closed surface is equal to $1/\epsilon_0$ times the total charge enclosed by the surface.

(ii) ***Maxwell's second equation*** expresses Gauss's law in magnetostatics, which states that the net magnetic flux through any closed surface is zero.

(iii) ***Maxwell's third equation*** is the Faraday's law of electromagnetic induction, which states that the induced electromotive force around any closed surface is equal to the negative time rate of change of the magnetic flux through the path enclosing the surface.

(iv) ***Maxwell's fourth equation*** represents the generalized form of Ampere's law as extended by Maxwell to account for the time-varying magnetic fields. It states that the magnetomotive force around a closed path is equal to the sum of conduction current and displacement current through the surface bounded by that path.

ELECTROMAGNETIC ENERGY (POYNTING THEOREM):

➢This theorem analyses the transportation of energy in the medium from one place t another due to the propagation of electromagnetic waves.

$$\int_V (\vec{E}\cdot\vec{J})\,dV = \frac{\partial}{\partial t}\int_V \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})\,dV - \int_s (\vec{E}\times\vec{H})\cdot d\vec{S}$$

- ➢ Above equation is known as ***Poynting theorem***. Each term of this expression has its own physical significance, which can be explained as follows:

(i) The term in L.H.S. is the generalised statement of Joule's law and represents the total power dissipated in volume *V*.

(ii) The first term on the right-hand side of the equation is the sum of energy stored in electric field 1/2(*E*.*D*) and in magnetic field 1/2(*B*.*H*) or the total energy stored in electromagnetic field. Therefore, this term represents the rate of change in energy stored in volume *V*.

- ➢ Thus, the ***Poynting theorem*** states that the work done on the charge by an electromagnetic force is equal to the decrease in energy stored in the field
- ➢ , less than the energy which flowed out through the surface.

CHARACTERISTIC IMPEDANCE OR INTRINSIC IMPEDANCE, OR WAVE IMPEDANCE OF THE FREE SPACE:

The ratio E_0 / H_0 has the dimensions of electric resistance and a universal constant called ***characteristic impedance*****, or *****intrinsic impedance*****, or** ***wave impedance*** of free space.

Or,

$$\frac{E}{H} = Z_0 = \frac{E_0}{H_0} = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} \left(c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \right)$$

$$\frac{E_0}{H_0} = \sqrt{\frac{4\pi \times 10^{-7}\,\mathrm{Wb / A\,m}}{8.85 \times 10^{-12}\,\mathrm{C^2 N m^2}}} = 376.72 = 377\,\Omega$$

ENERGY FLOW IN PLANE ELECTROMAGNETIC WAVE

For a plane electromagnetic wave, the energy flow per unit time per unit area is given by Poynting vector as:

$$\vec{S} = \vec{E} \times \vec{H}$$

Writing the value of H in terms of electric field and doing some mathematical calculation we can write following expression

$$\vec{S} = \frac{E^2}{Z_0} \hat{n}$$

For a complete cycle, the average value of S is given as

$$< \vec{S} > = \frac{1}{Z_0} < E^2 > \hat{n}$$

$$= \frac{1}{Z_0} R_e \left[E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right]^2 \hat{n}$$

$$= \frac{1}{Z_0} < \left[E_0 \cos(\vec{k} \cdot \vec{r} - \omega t) \right]^2 > \hat{n} \qquad (e^{i\theta} = \cos\theta + i \sin\theta)$$

$$= \frac{1}{Z_0} E_0^2 \cdot \frac{1}{2} \hat{n} \qquad \left(< \cos^2\theta > = \frac{1}{2} \right)$$

or

$$< \vec{S} > = \frac{E_{rms}^2}{Z_0} \hat{n}$$

ENERGY DENSITY IN PLANE ELECTROMAGNETIC WAVE IN FREE SPACE

The energy per unit volume, or energy density in an electric field E, is given by

$$U_E = \frac{1}{2} \in_0 E^2$$

and the energy per unit volume, or energy density in a magnetic field B, is given by

$$U_B = \frac{1}{2} \mu_0 H^2$$

In an electromagnetic field, both E and B are present.

Therefore, the electromagnetic energy density is given as

$$U = U_A + U_B = \frac{1}{2} \in_0 E^2 + \frac{1}{2} \mu_0 H^2$$

But in free space, we have

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\in_0}}$$

or $$H = E \sqrt{\frac{\in_0}{\mu_0}}$$

$\therefore$ $$U = \frac{1}{2} \in_0 E^2 + \frac{1}{2} \in_0 E^2$$

Therefore, the average energy density per unit time is

$$< U > = < \epsilon_0 E^2 > = \epsilon_0 < \left(E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right)^2_{\text{real}} >$$

$$= \epsilon_0 E_0^2 < \cos^2(\vec{k} \cdot \vec{r} - \omega t) >$$

$$= \frac{\epsilon_0 E_0^2}{2} \qquad \left(< \cos^2 \theta > = \frac{1}{2} \right)$$

or $$< U > = \epsilon_0 E^2_{\text{rms}}$$

Dividing Eq. (17.37) by Eq. (17.40), we get

$$\frac{< \vec{S} >}{< \vec{U} >} = \frac{\hat{n}}{\epsilon_0 Z_0} = \frac{\hat{n}}{\epsilon_0 \sqrt{\mu_0 / \epsilon_0}} = \frac{\hat{n}}{\sqrt{\mu_0 / \epsilon_0}} = c\hat{n}$$

$$< \vec{S} > = c < U > \hat{n}$$

or energy flux = velocity of light × energy density.

ELECTROMAGNETIC WAVE IN FREE SPACE AND ITS SOLUTION

For free space or vacuum, Maxwell's equations are as follows:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{17.42}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{17.43}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{17.44}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \in_0 \frac{\partial \vec{E}}{\partial t} \tag{17.45}$$

Taking the curl of Eq. (17.44), we get

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

Using the identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\,\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$, we get

$$(\vec{\nabla} \cdot \vec{E})\vec{\nabla} - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$0 - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \qquad \text{[using Eq. (17.42)]}$$

or

$$\nabla^2 \vec{E} = \frac{\partial}{\partial t} \left(\mu_0 \in_0 \frac{\partial \vec{E}}{\partial t} \right) \qquad \text{[using Eq. (17.45)]}$$

or $$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad (17.46)$$

Equation (17.46) is a wave equation for electric field in free space. Similarly, for magnetic field, we can have

$$\nabla^2 B = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2} \qquad (17.47)$$

Now, the equation of wave propagating with a velocity υ is given as

$$\nabla^2 y = \frac{1}{\upsilon^2} \frac{\partial^2 y}{\partial t^2} \qquad (17.48)$$

Comparison of Eqs. (17.46) or (17.47) with Eq. (17.48) gives the velocity of propagation of electric and magnetic vectors in free space.

i.e., $$\frac{1}{\upsilon^2} = \mu_0 \epsilon_0$$

or $$\upsilon = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

or $$\upsilon = \frac{1}{\sqrt{\frac{\mu_0}{4\pi} \times 4\pi\epsilon_0}} = \frac{1}{\sqrt{\frac{\mu_0}{4\pi}}} \times \sqrt{\frac{1}{4\pi\epsilon_0}}$$

Now, $$\frac{\mu_0}{4\pi} = 10^{-7} \text{ Wb/A-m and } \frac{1}{4\pi\epsilon_0} = 9 \times 10^9$$

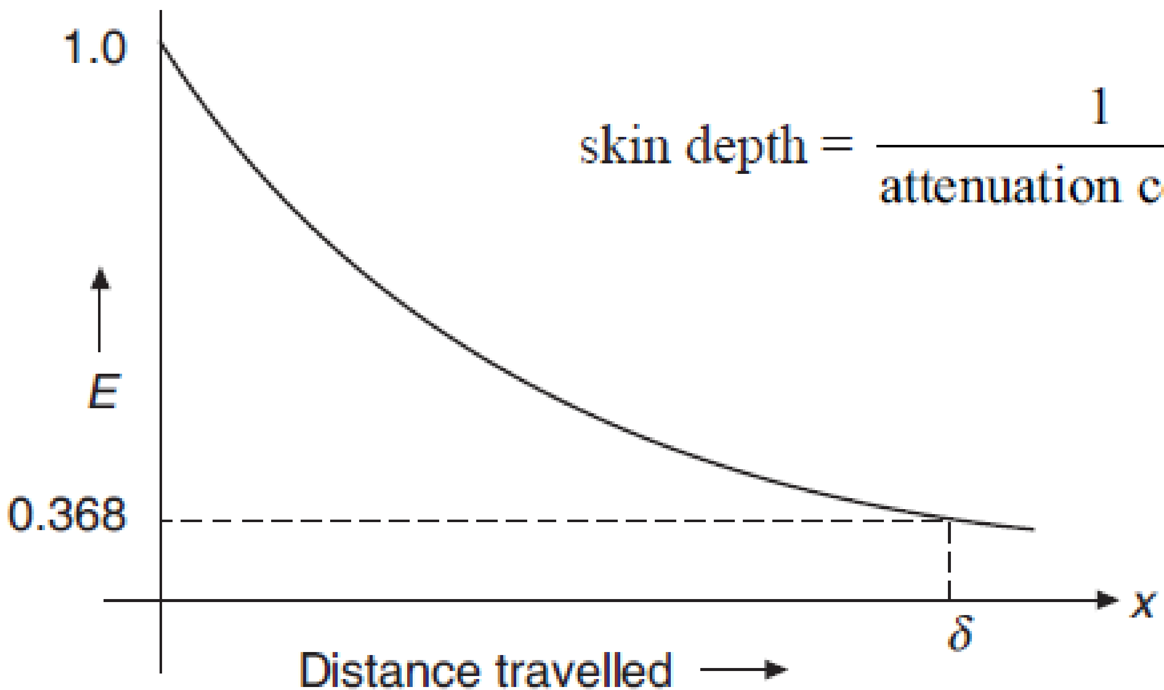
$\therefore$ $$\upsilon = \sqrt{\frac{1}{10^{-7}}} \times \sqrt{9 \times 10^9}$$

$$\upsilon = 3 \times 10^8 \text{ m/s} = c \text{ (velocity of light)}$$

Hence, the electromagnetic waves propagate in free space with the velocity of light.

DEPTH OF PENETRATION: SKIN DEPTH:

Skin depth describes the conducting behaviour in electromagnetic field, and in ra
communication, it is defined as the depth for which the strength of electric field associated with the electromagnetic wave reduces to **1/*e* times** of its initial value.



➢For good conductors, we have -

$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

Therefore, the skin depth is -

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Or,

$$\delta = \sqrt{\frac{1}{f\mu\sigma\pi}}$$

Thus, the skin depth decreases with the increase in frequency.

➢However, for poor conductors or good dielectrics, or insulators, the skin depth may be given as

$$\delta = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$$

Thus, for dielectrics, the skin depth is independent of fequency