Displacement Current

Another step towards
Maxwell's Equations

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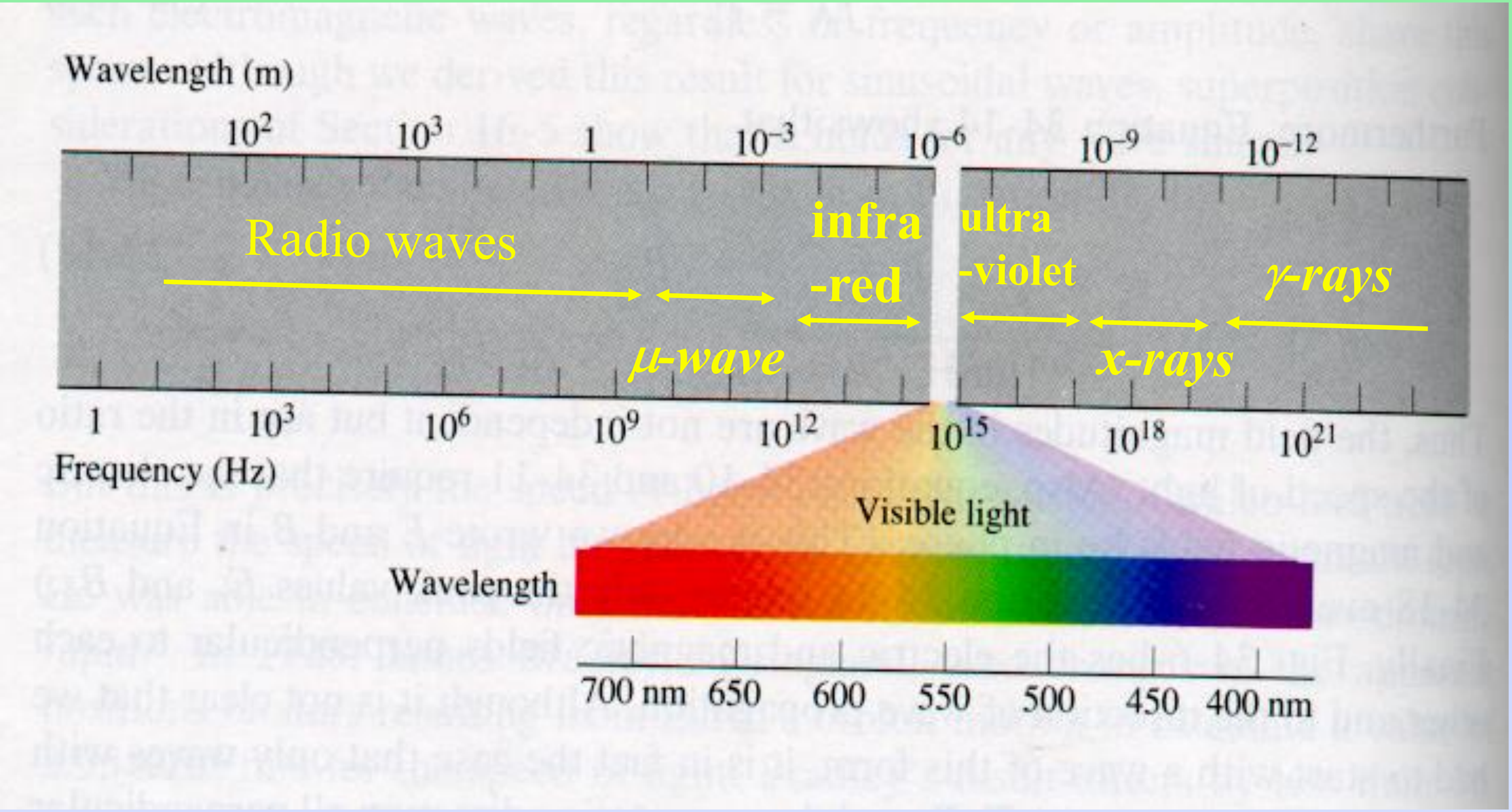
The Electromagnetic Spectrum



The Equations of Electromagnetism
(at this point …)

Gauss' Law for Electrostatics

$$\oint \underline{E} \bullet \underline{dA} = \frac{q}{\varepsilon_0}$$

Gauss' Law for Magnetism

$$\oint \underline{B} \bullet \underline{dA} = 0$$

Faraday's Law of Induction

$$\oint \underline{E} \bullet \underline{dl} = -\frac{d\Phi_B}{dt}$$

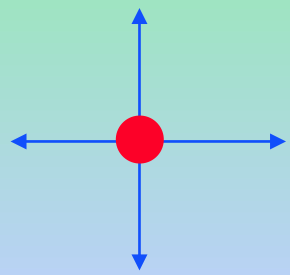
Ampere's Law

$$\oint \underline{B} \bullet \underline{dl} = \mu_0 I$$

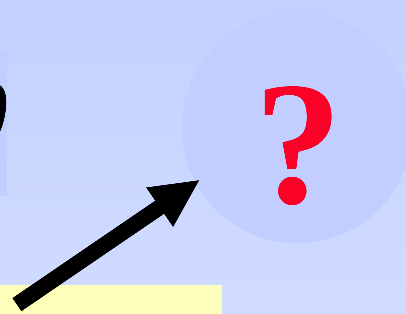
The Equations of Electromagnetism

Gauss's Laws ***..monopole..***

1 $$\oint \underline{E} \bullet d\underline{A} = \frac{q}{\varepsilon_0}$$



2 $$\oint \underline{B} \bullet d\underline{A} = 0$$



...there's no magnetic monopole....!!

The Equations of Electromagnetism

Faraday's Law

3 $$\oint \underline{E} \bullet \underline{dl} = -\frac{d\Phi_B}{dt}$$

.. if you change a magnetic field you induce an electric field.........

Ampere's Law

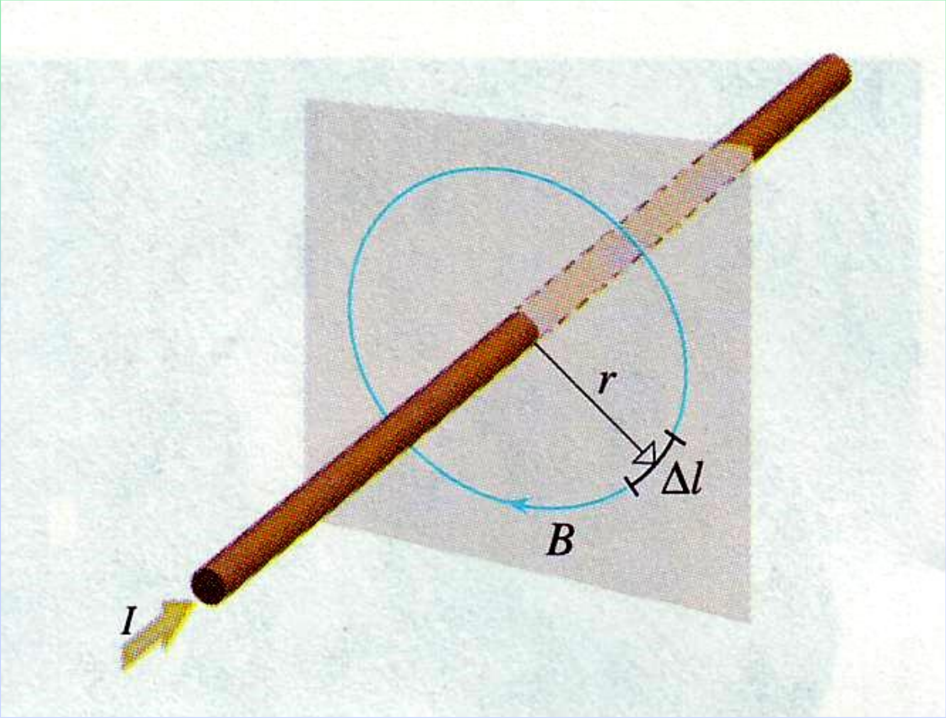
4 $$\oint \underline{B} \bullet \underline{dl} = \mu_0 I$$

.......is the reverse true..?

Basic Definition of Current

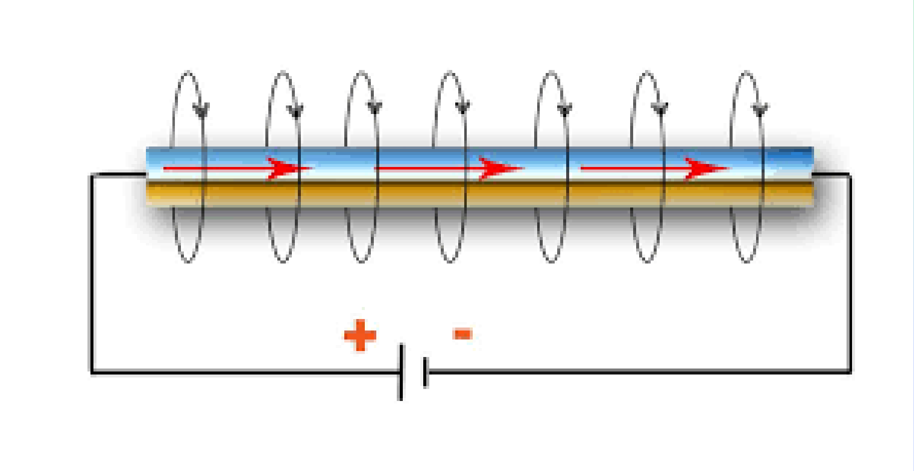
- $I = neAv_d$
- For current flowing through a conductor it is must that the electronic charge should move through it.
- Now the question is how to test the flow of current through a conductor in the simplest way ??
- For it let us recall Ampere's law….

Recall Ampere's Law



$$\oint \vec{B} \cdot dl = \mu_0 I_{enc}$$

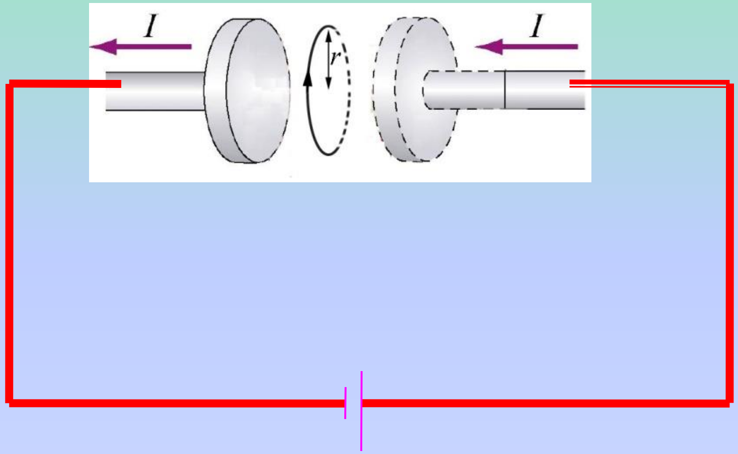
View of Magnetic field around a current carrying conductor



- ➢The presence of electric current can be observed using the magnetic compass

- ➢When we will place the magnetic compass around the current carrying conductor there will be deflection in it.

What will happen if any where there is no current carrying conductor in a circuit

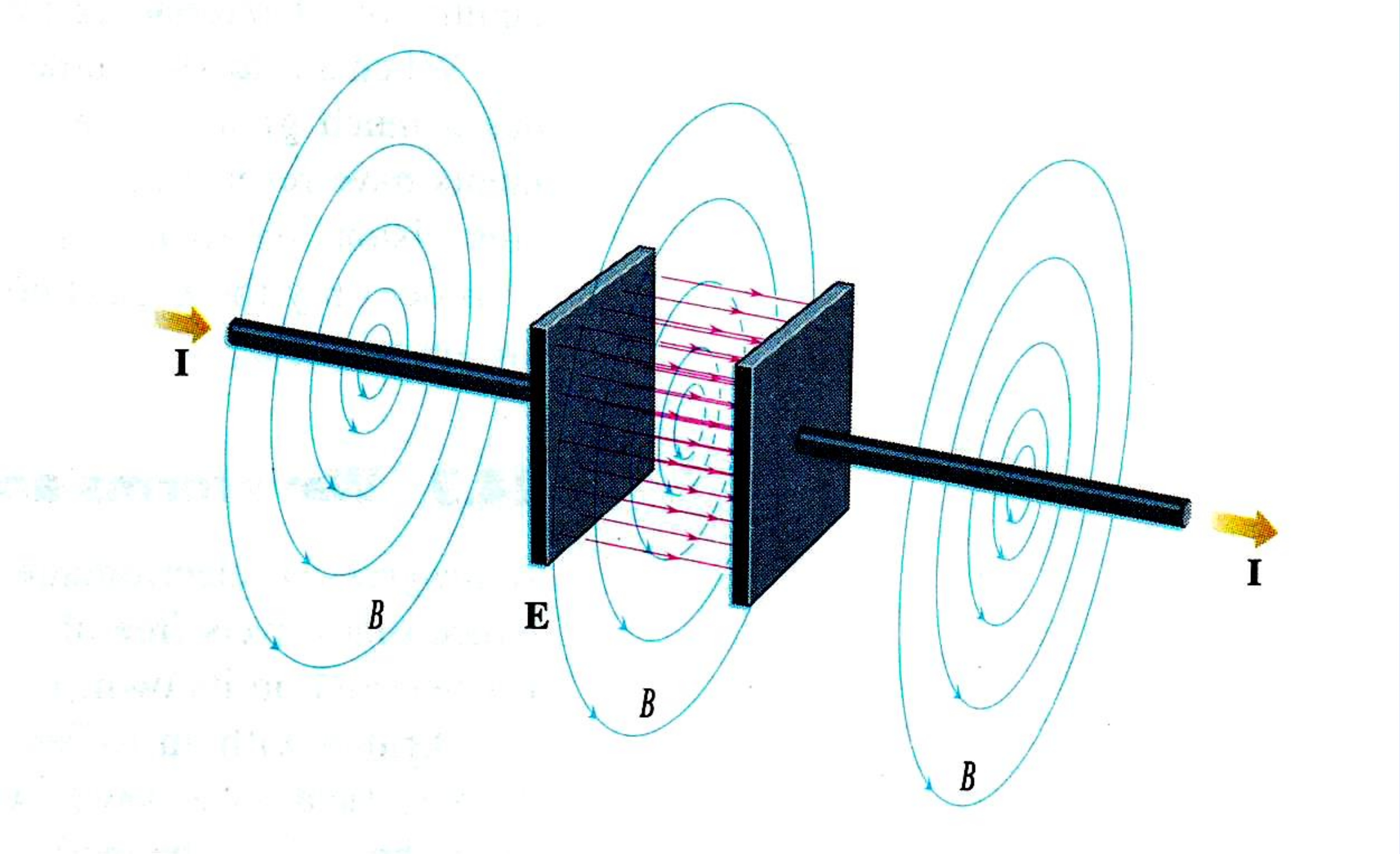


➢The current flowing in the circuit is I, but what is the current flowing between the plates of capacitor.

➢Obviously Zero !!!

- Now according to the Ampere’s law there should be no magnetic field between the plates of the capacitor.
- Let us verify it experimentally what the situation is prevailing between the plates of a capacitor, by putting magnetic compass there.
- Surprisingly !! There is the deflection in the magnetic compass.
- It suggests that the definition of current what we have read is either wrong or Ampere’s law need modification for its generalization .
- This important task was accomplished by Maxwell on the basis of change in electrical field between plates of capacitor and introducing the concept of displacement current.

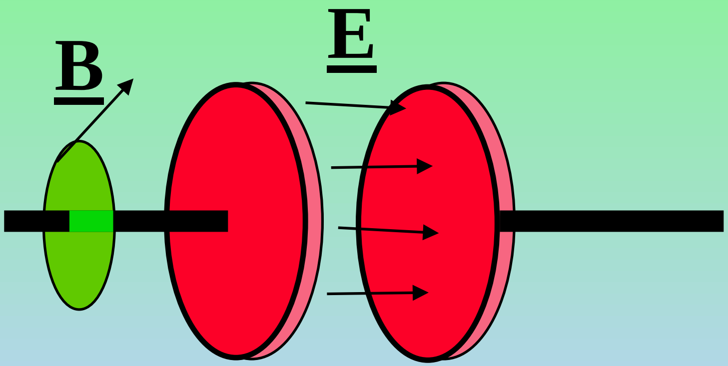
➢If Ampere's Law still holds, there must be a magnetic field generated by the changing E-field between the plates. This induced B-field makes it look like there is a current (call it the **displacement current**) passing through the plates.



...lets take a look at charge flowing into a capacitor...

...when we derived Ampere's Law we assumed constant current...

$$\oint \underline{B} \bullet \underline{dl} = \mu_0 I$$



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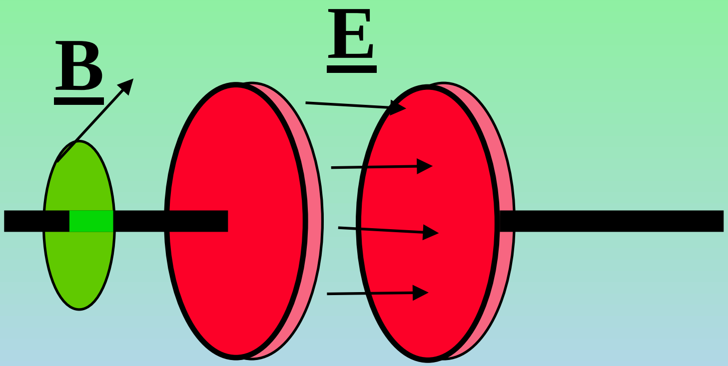
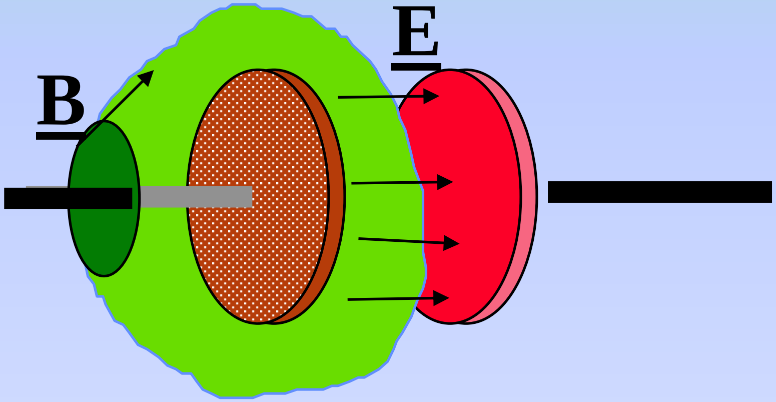
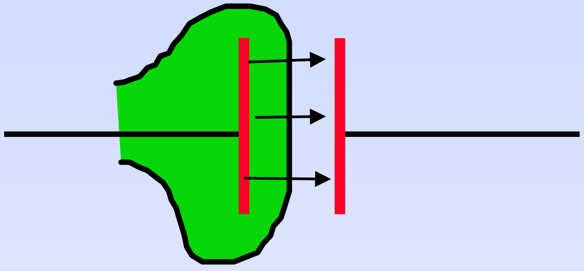


.. if the loop encloses one plate of the capacitor..there is a problem … I = 0

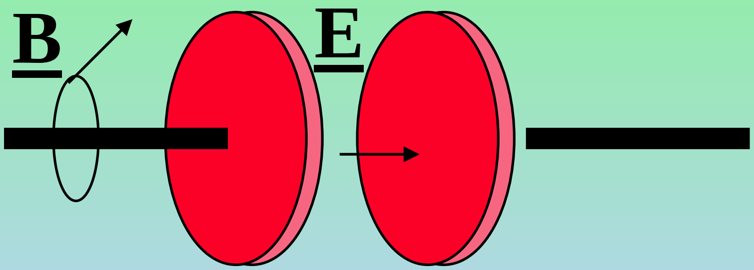


Side view: (Surface is now like a bag:)



Maxwell solved this problem by realizing that....

Inside the capacitor there must be an induced magnetic field...



How?.

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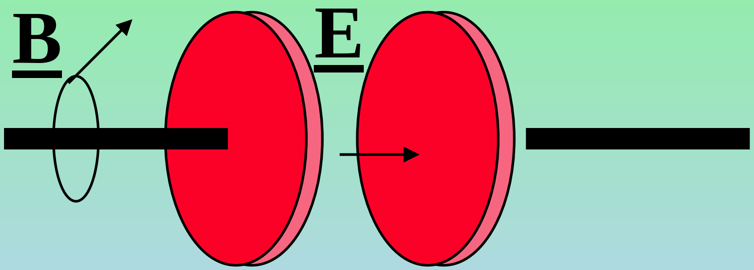
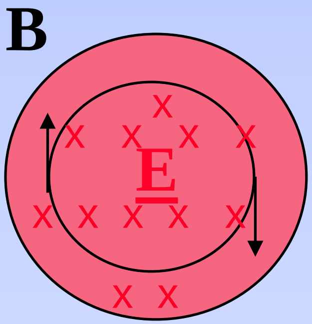


How?. Inside the capacitor there is a changing E ⇒



A changing electric field induces a magnetic field

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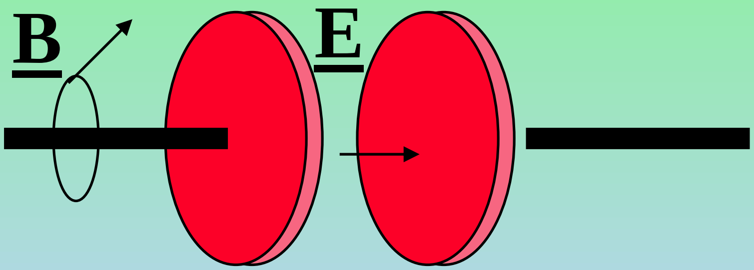

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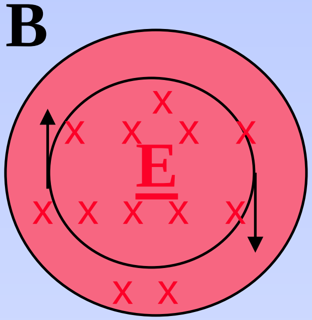

A changing electric field induces a magnetic field

$$\oint \underline{B} \bullet \underline{dl} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 I_d$$

where I_d is called the ***<u>displacement current</u>***

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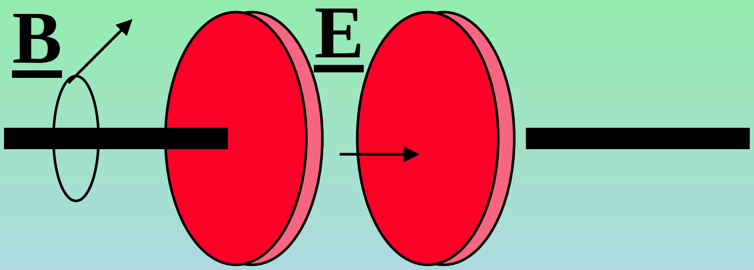

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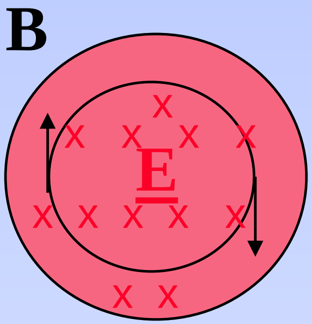

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where I_d is called the <u>displacement current</u>

Therefore, Maxwell's revision of Ampere's Law becomes....

$$\oint \underline{B} \bullet \underline{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Derivation of Displacement Current

For a capacitor, $q = \varepsilon_0 EA$ and $I = \dfrac{dq}{dt} = \varepsilon_0 \dfrac{d(EA)}{dt}$.

Now, the electric flux is given by EA, so: $I = \varepsilon_0 \dfrac{d(\Phi_E)}{dt}$,
where this current , not being associated with charges, is called the "Displacement current", I_d.

Hence: $$I_d = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

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Hence: $$I_d = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

and: $$\oint \underline{B} \bullet \underline{dl} = \mu_0 (I + I_d)$$

$$\Rightarrow \oint \underline{B} \bullet \underline{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Maxwell's Equations of Electromagnetism

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Ampere's Law

$$\oint \underline{B} \bullet \underline{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Maxwell's Equations of Electromagnetism in Vacuum (no charges, no masses)

Consider these equations in a vacuum.....

......no mass, no charges. no currents.....

$$\oint \underline{E} \bullet \underline{dA} = \frac{q}{\varepsilon_0} \longrightarrow \oint \underline{E} \bullet \underline{dA} = 0$$

$$\oint \underline{B} \bullet \underline{dA} = 0 \longrightarrow \oint \underline{B} \bullet \underline{dA} = 0$$

$$\oint \underline{E} \bullet \underline{dl} = -\frac{d\Phi_B}{dt} \longrightarrow \oint \underline{E} \bullet \underline{dl} = -\frac{d\Phi_B}{dt}$$

$$\oint \underline{B} \bullet \underline{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \longrightarrow \oint \underline{B} \bullet \underline{dl} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Maxwell's Equations of Electromagnetism in Vacuum (no charges, no masses)

$$\oint \underline{E} \bullet d\underline{A} = 0$$

$$\oint \underline{B} \bullet d\underline{A} = 0$$

$$\oint \underline{E} \bullet d\underline{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \underline{B} \bullet d\underline{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Electromagnetic Waves

Faraday's law: **dB/dt ⟶ electric field**

Maxwell's modification of Ampere's law

dE/dt ⟶ magnetic field

$$\oint \underline{B} \bullet \underline{dl} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

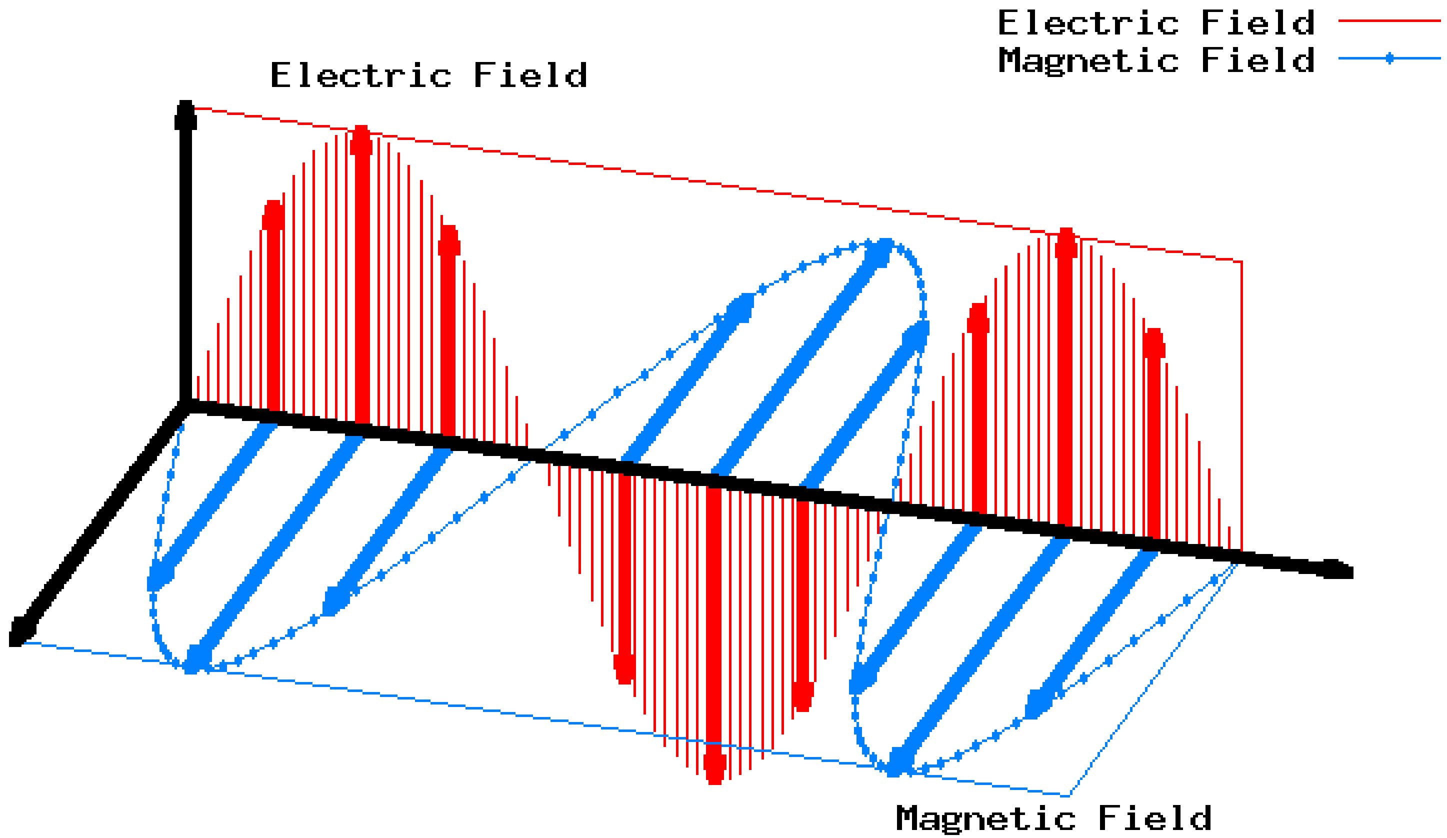
$$\oint \underline{E} \bullet \underline{dl} = -\frac{d\Phi_B}{dt}$$

These two equations can be solved simultaneously.

The result is:

$$\underline{\mathbf{E}}(x, t) = E_P \sin(kx - \omega t)\ \hat{j}$$

$$\underline{\mathbf{B}}(x, t) = B_P \sin(kx - \omega t)\ \hat{k}$$


Electric Field
Magnetic Field
Electric Field
Magnetic Field

Electromagnetic Waves

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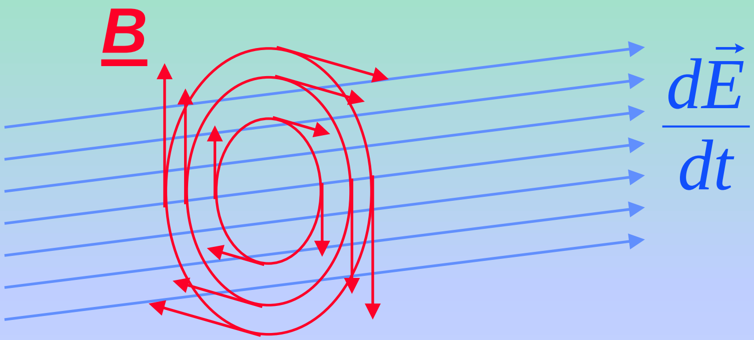

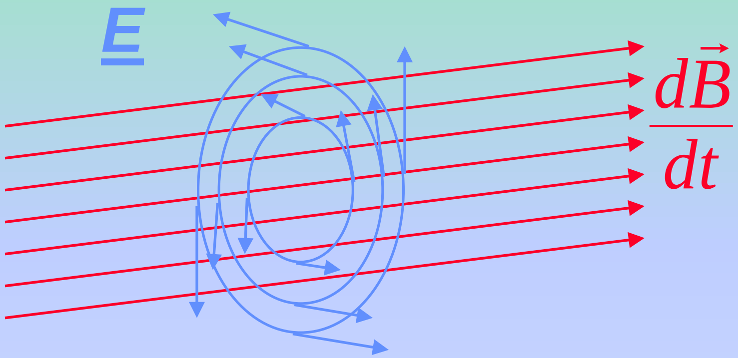

Electromagnetic Waves

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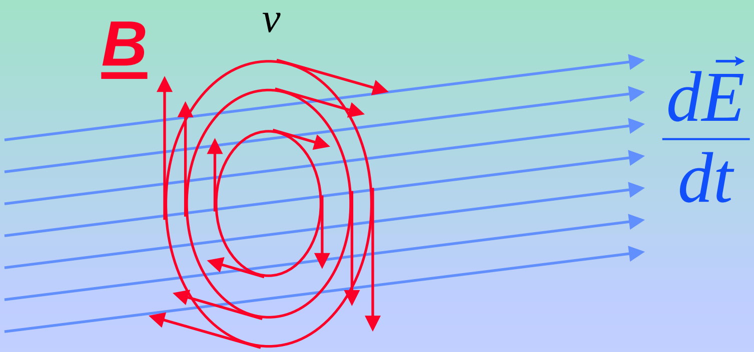
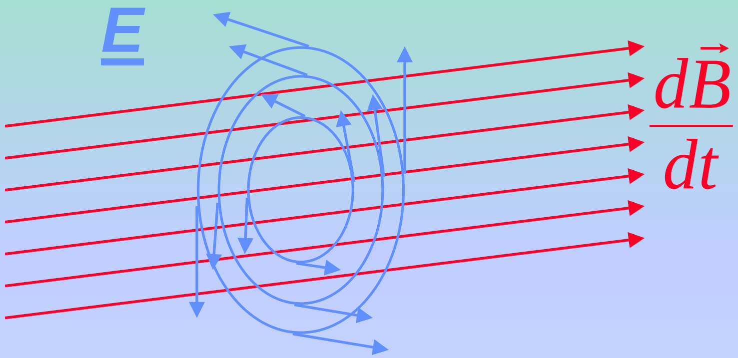




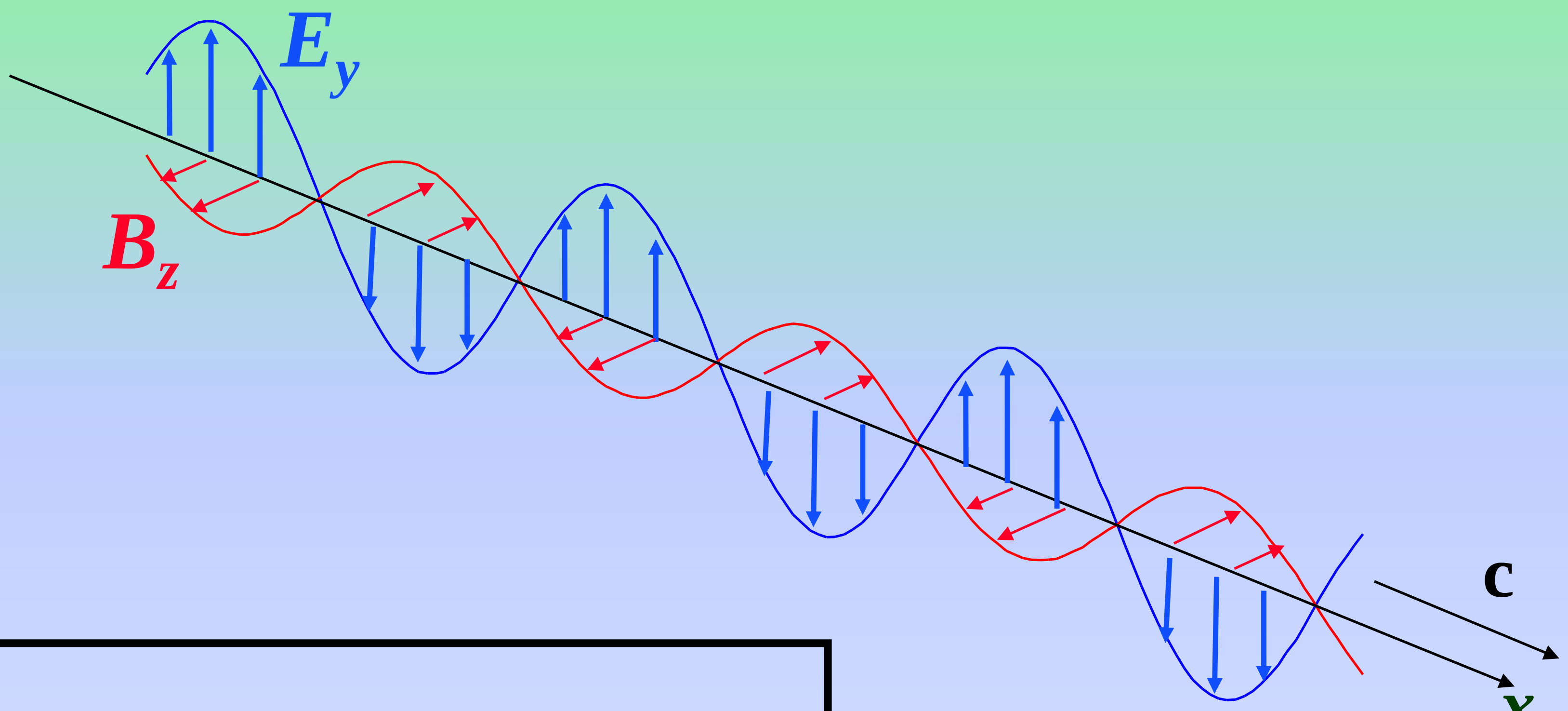
Special case..PLANE WAVES... $\vec{E} = E_y(x,t)\hat{j}$ $\quad \vec{B} = B_z(x,t)\hat{k}$

satisfy the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

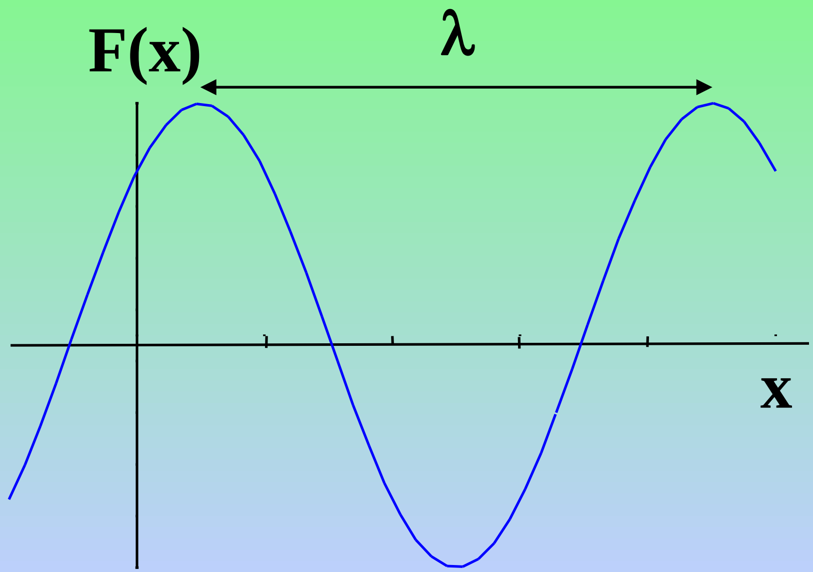
Maxwell's Solution → $\psi = A \sin(\omega t + \phi)$

Plane Electromagnetic Waves



$\underline{E}(x, t) = E_P \sin(kx - \omega t)\ \hat{j}$

$\underline{B}(x, t) = B_P \sin(kx - \omega t)\ \hat{k}$



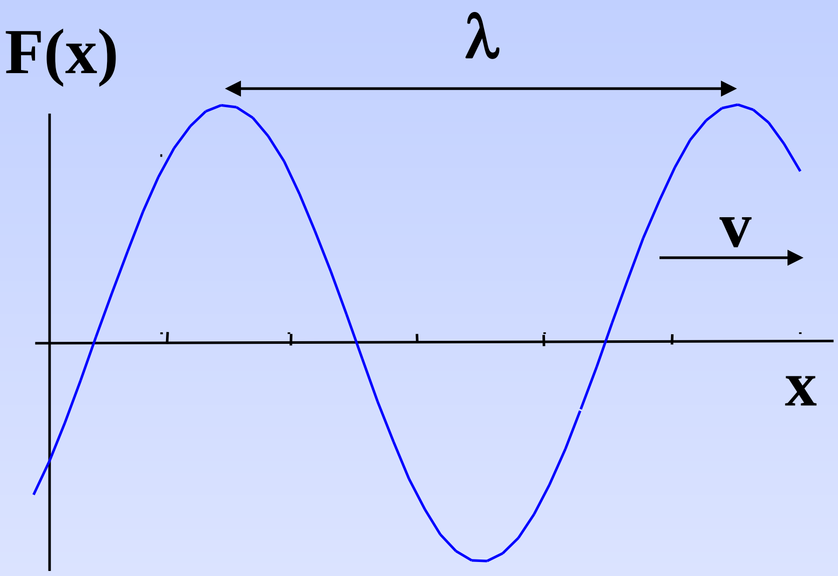
Static wave

$F(x) = F_P \sin(kx + \phi)$

$k = 2\pi / \lambda$

k = wavenumber

λ = wavelength



Moving wave

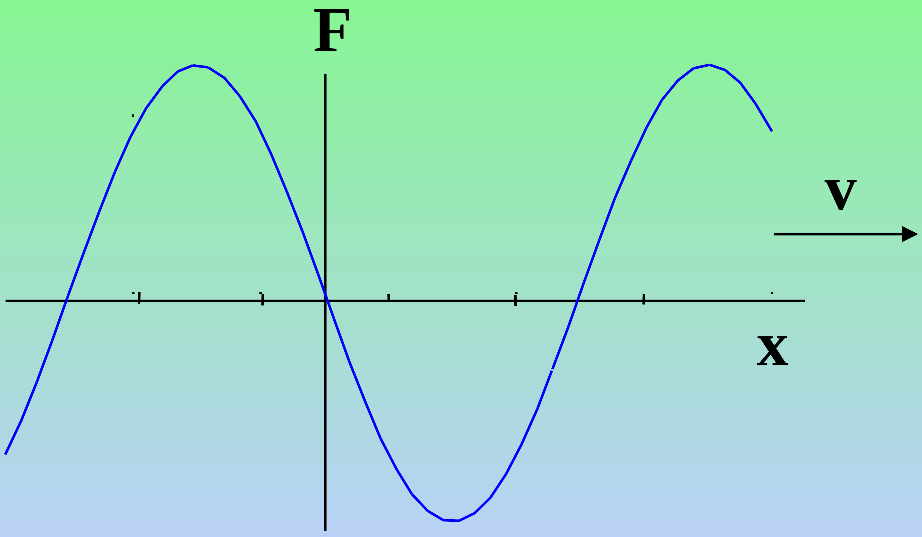
$F(x, t) = F_P \sin(kx - \omega t)$

$\omega = 2\pi / f$

ω = angular frequency

f = frequency

$v = \omega / k$



Moving wave

$$F(x, t) = F_P \sin (kx - \omega t)$$

What happens at x = 0 as a function of time?

$F(0, t) = F_P \sin(-\omega t)$

For $x = 0$ and $t = 0 \Rightarrow F(0, 0) = F_P \sin (0)$

For $x = 0$ and $t = t \Rightarrow F(0, t) = F_P \sin (0 - \omega t) = F_P \sin (- \omega t)$

This is equivalent to: $kx = - \omega t \Rightarrow x = - (\omega/k)\, t$

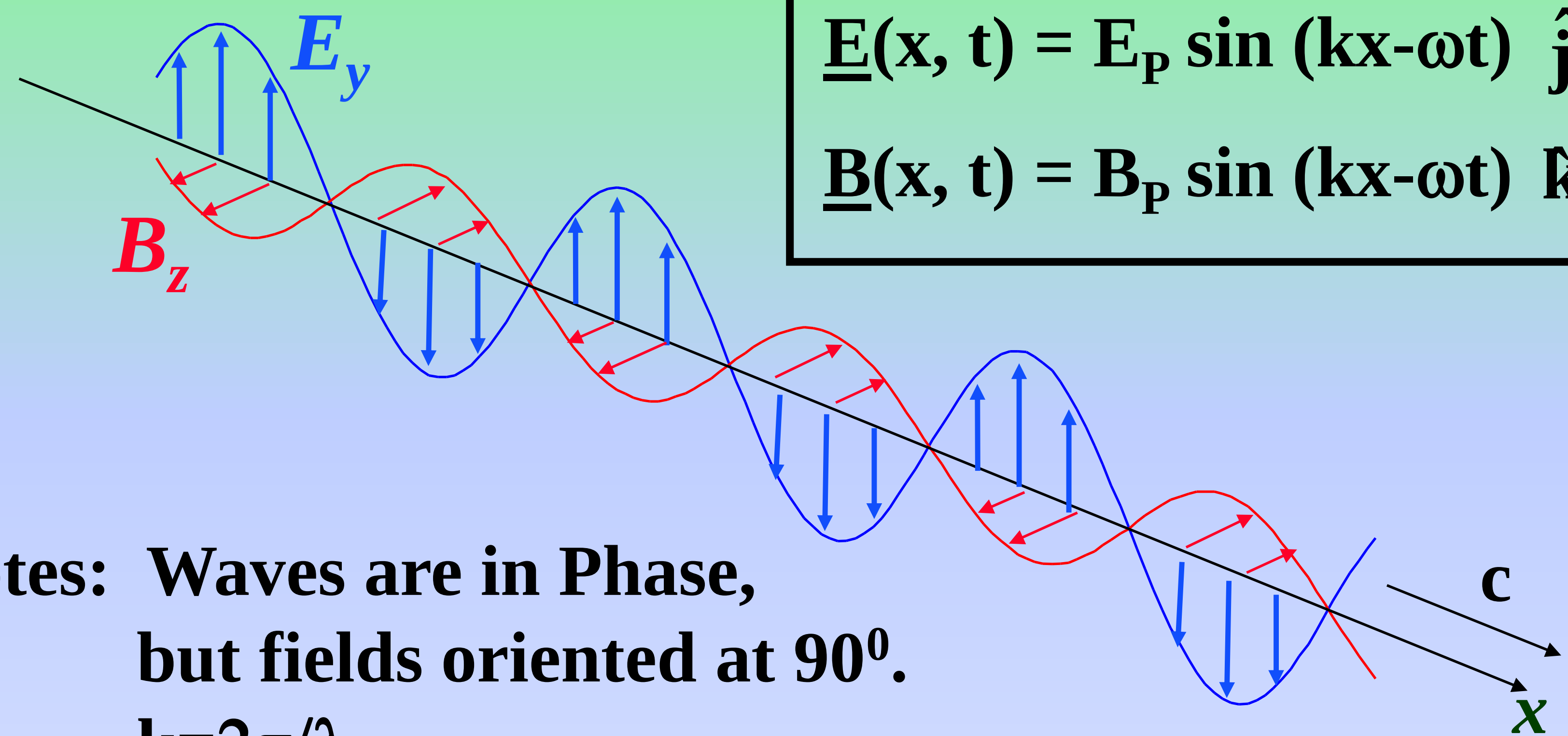
F(x=0) at time t is the same as F[x=-(ω/k)t] at time 0

The wave moves to the right with speed ω/k

Plane Electromagnetic Waves

$\underline{\mathbf{E}}(x, t) = E_P \sin(kx - \omega t)\ \hat{j}$

$\underline{\mathbf{B}}(x, t) = B_P \sin(kx - \omega t)\ \hat{k}$



Notes: Waves are in Phase,
but fields oriented at 90^0.
$k=2\pi/\lambda$.
Speed of wave is $c=\omega/k$ (= $f\lambda$)

$$c = 1/\sqrt{\varepsilon_0\mu_0} = 3\times10^8\, m/s$$

Deduction of Maxwell's Laws in Differential form

Gauss Divergence Theorem (Relation between Surface and Volume Integration)

This theorem states that the flux of a vector field $\vec{F}$, over any closed surface S, is equal to the volume integral of the divergence of that vector field over the volume V enclosed by the surface S.

$$\int_S \vec{F} \cdot d\vec{S} = \int_V \text{div}\, \vec{F}\, dV$$

Stokes Theorem (Relation between Line Integral and Surface Integration)

This theorem states that the surface integral of the curl of a vector field $\vec{A}$, taken over any surface S, is equal to the line integral of $\vec{A}$ around the closed curve forming the periphery of the surface.

$$\iint_s (\text{Curl}\, \vec{A}) \cdot d\vec{S} = \oint \vec{A} \cdot \vec{dl}$$

$$\iint_s (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint \vec{A} \cdot \vec{dl}$$

1. Maxwell's first equation, div D = ρ or $\vec{\nabla} \cdot \vec{D} = \rho$:

➢When a dielectric is placed in a uniform electric field, its molecules get polarised. Thus, a dielectric in an electric field contains two types of charges—free charges, which are embedded, and polarisation charges or bound charges.

➢If ρ and ρ_P are the free and bound charge densities, respectively, at a point in a small volume element dv, then for such a medium, Gauss's law may be expressed as

$$\int_s \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_V (\rho + \rho_P)\, dV$$

where ϵ_0 is the permittivity of the free space.

Now, the bound charge density

$\rho_p = -\operatorname{div} \vec{P}$, where $\vec{P}$ is electric polarisation.

Therefore, $$\int_s \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_V (\rho - \operatorname{div} \vec{P})\, dV$$

$$\int_s \vec{E}\cdot d\vec{S} = \int_v \text{div}\, E\, dV = \frac{1}{\epsilon_0}\int_v \rho\, dV - \frac{1}{\epsilon_0}\int_v \text{div}\, \vec{P}\, dV$$

or $$\int_v \epsilon_0\, \text{div}\, \vec{E}\, dV + \int_v \text{div}\, \vec{P}\, dV = \int_v \rho\, dV$$

$$\int_v \text{div}\epsilon_0\, \vec{E}\, dV + \int_v \text{div}\, \vec{P}\, dV = \int_v \rho\, dV$$

$$\int_v \text{div}(\epsilon_0\, \vec{E} + \vec{P})\, dV = \int_v \rho\, dV$$

But $\epsilon_0 \vec{E} + \vec{P} = \vec{D}$ is the electric displacement vector.

Thus, $$\int_v \text{div}\, \vec{D}\, dV = \int_v \rho\, dV$$

or $$\int_v (\text{div}\, \vec{D} - \rho)\, dV = 0$$

Therefore, for an arbitrary surface, we have

$$\text{div}\ \vec{D} - \rho = 0$$

or $$\text{div}\ \vec{D} = \rho$$

or $$\vec{\nabla}\cdot\vec{D} = \rho$$

2. *Maxwell's second equation, div* $\vec{B} = 0$ *or* $\vec{\nabla} \cdot \vec{B} = 0$:

➢It has been experimentally observed that the number of magnetic lines of force entering any closed surface enclosing a volume is exactly the same as that leaving it, i.e., the net magnetic flux through any closed surface is always zero.

Hence,

$$\phi_B = \oint_S \vec{B} \cdot d\vec{S} = 0$$

The above expression implies that a monopole or an isolated magnetic pole cannot exist to serve as a source or sink for the line of magnetic induction $\vec{B}$. This expression is also known as *Gauss's law in magnetostatics*.

Using Gauss divergence theorem in Eq. (17.6), we have

$$\oint \vec{B} \cdot d\vec{S} = \int_V \text{div}\,\vec{B}\,dV = 0$$

where V is the volume enclosed by surface S.

Hence, for an arbitrary surface,

$$\text{div}\,\vec{B} = 0$$

or $$\vec{\nabla} \cdot \vec{B} = 0$$

3. Maxwell's third equation (Faraday's law of electromagnetic induction):

➢According to Faraday's law of electromagnetic induction, the induced emf around a closed circuit is equal to the negative time rate of change of magnetic flux linked with the circuit, i.e.

$$e = -\frac{d\phi_B}{dt}$$

If $\vec{B}$ is the magnetic induction, then the magnetic flux linked with an area $d\vec{S}$ is

$$\phi_B = \int_s \vec{B} \cdot d\vec{S}$$

On combining Eqs. (17.18) and (17.19), we get

$$e = -\frac{d}{dt}\int_s (\vec{B} \cdot d\vec{S})$$

or

$$e = \int_s \frac{\partial}{\partial t}(\vec{B} \cdot d\vec{S})$$

According to definition, the induced emf is related to the corresponding electric field as

$$e = \int_c \vec{E} \cdot d\vec{l}$$

Equations (17.20) and (17.21) will give

$$\int_c \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial}{\partial t}(\vec{B} \cdot d\vec{S})$$

$$= -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

Now, using Stoke's theorem on left-hand side, we get

$$\int_c \vec{E} \cdot d\vec{l} = \int_s \text{curl } \vec{E} \cdot d\vec{S}$$

Thus, we have

$$\int_s \text{curl } \vec{E} \cdot d\vec{S} = \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

or
$$\int_s \left(\text{curl } \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{S} = 0$$

For any arbitrary surface *dS*, we will have

$$\text{curl } \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

or
$$\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

i.e.,
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

4. Maxwell's fourth equation (modified Ampere's law):

From the Ampere's law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

Using formula $I = \oint \vec{J} \cdot d\vec{S}$ $\left(\text{using } J = \frac{I}{A}\right)$

we get

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \oint \vec{J} \cdot d\vec{S}$$

Using Stoke's theorem on the left-hand side of the above expression, we get

$$\oint_s \text{curl } \vec{B} \cdot d\vec{S} = \mu_0 \oint_s \vec{J} \cdot d\vec{S}$$

$$\frac{1}{\mu_0} \oint_s \text{curl } \vec{B} \cdot d\vec{S} = \oint_s \vec{J} \cdot d\vec{S}$$

$$\oint_s \text{curl } \frac{\vec{B}}{\mu_0} \cdot d\vec{S} = \oint_s \vec{J} \cdot d\vec{S}$$

Now, from dielectric properties, we have

$$\frac{B}{\mu_0} = H$$

$\therefore$ $$\int_s \text{curl } \vec{H} \cdot d\vec{S} = \int_s \vec{J} \cdot d\vec{S}$$

or $$\int_s (\text{curl } \vec{H} - \vec{J}) \cdot d\vec{S} = 0$$

For an arbitrary surface, we have

$$\text{curl } \vec{H} - \vec{J} = 0$$

or

$$\text{curl } \vec{H} = \vec{J}$$

Taking divergence on both sides, we get

$$\text{div curl } \vec{H} = \text{div } \vec{J}$$

But div curl $\vec{H} = 0$ (From vector calculus)

$$\therefore \quad \text{div } \vec{J} = 0$$

From continuity equation, we have

$$\text{div } \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Hence,

$$\frac{\partial \rho}{\partial t} = 0$$

or

$$\rho = \text{constant (static)}$$

This implies that Ampere's law is applicable only for static charges. However, for time-varying fields, Maxwell suggested that Ampere's law must be modified by adding a quantity having dimension as that of current and produced due to polarisation of charges. This physical quantity is called displacement current (Jd).

Thus, modified Ampere's law now becomes

$$\text{curl } \vec{H} = \vec{J} + \vec{J}_d$$

Taking divergence on both sides, we get

$$\text{div curl } \vec{H} = \text{div } \left(\vec{J} + \vec{J}_d\right)$$

$$0 = \text{div } \vec{J} + \text{div } \vec{J}_d$$

or $$\text{div } \vec{J} = -\text{ div } \vec{J}_d$$

But $$\text{div } \vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{(Continuity equation)}$$

$\therefore$ $$\text{div } \vec{J}_d = \frac{\partial \rho}{\partial t}$$

But $$\rho = \text{div } \vec{D} \, *$$

$\therefore$ $$\text{div } \vec{J}_d = \frac{\partial}{\partial t}(\text{div } \vec{D})$$

$$= \text{div } \left(\frac{\partial \vec{D}}{\partial t}\right)$$

or $$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

Therefore, modified Ampere's law now becomes

$$\text{curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

CONCLUSIONS

- MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM
- MAXWELL'S EQUATIONS IN INTEGRAL FORM

MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM:

(i) $\vec{\nabla} \cdot \vec{D} = \rho$ or $\text{Div } \vec{D} = \rho$

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ or $\text{Div } \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ or $\text{Curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ or $\text{Curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

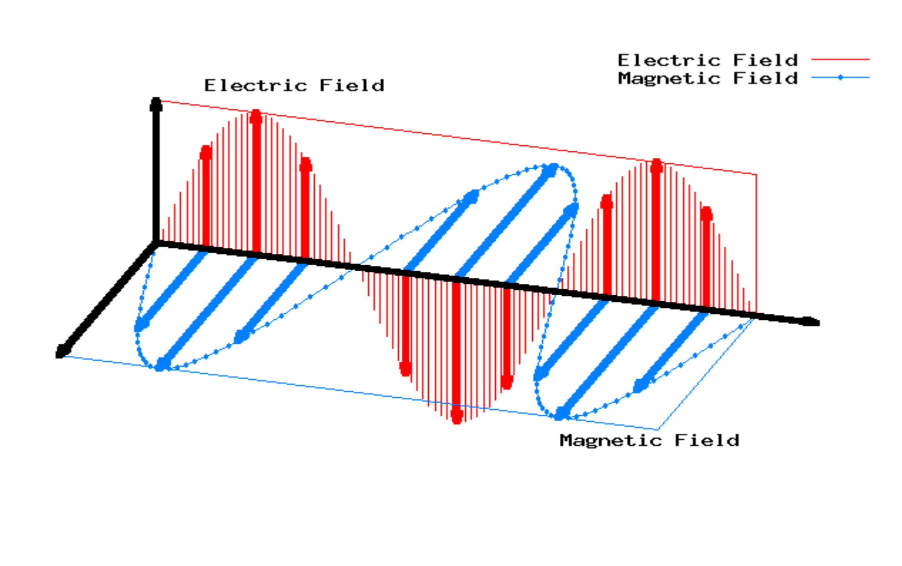
MAXWELL'S EQUATIONS IN INTEGRAL FORM:

(i) $\int_s \vec{D} \cdot \vec{d}S = \int_V \rho dV$ or $\oint_s \vec{E} \cdot \vec{d}S = q$

(ii) $\oint_s \vec{B} \cdot \vec{d}S = 0$

(iii) $\oint \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \int \vec{B} \cdot \vec{dS}$

(iv) $\oint \vec{H} \cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{dS}$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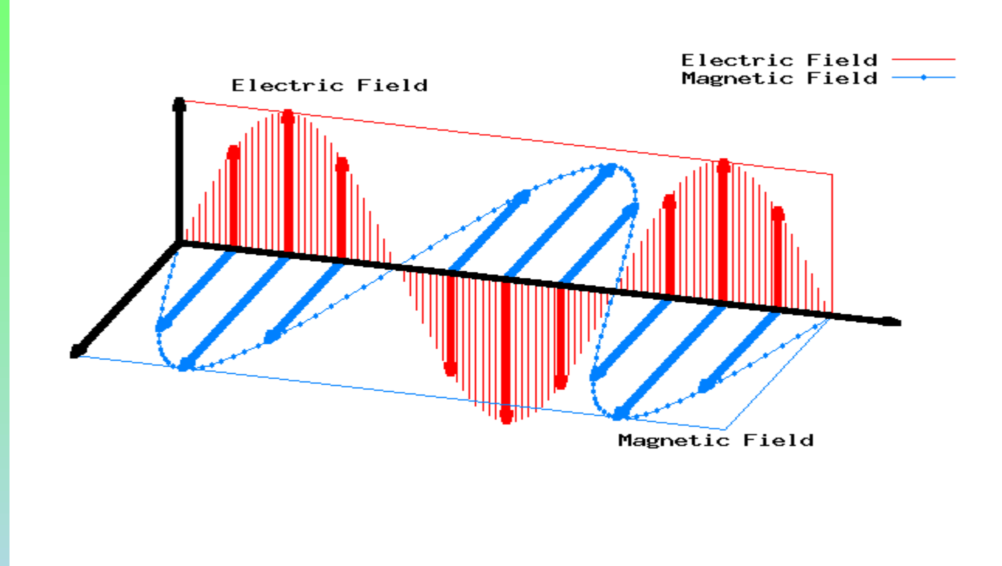
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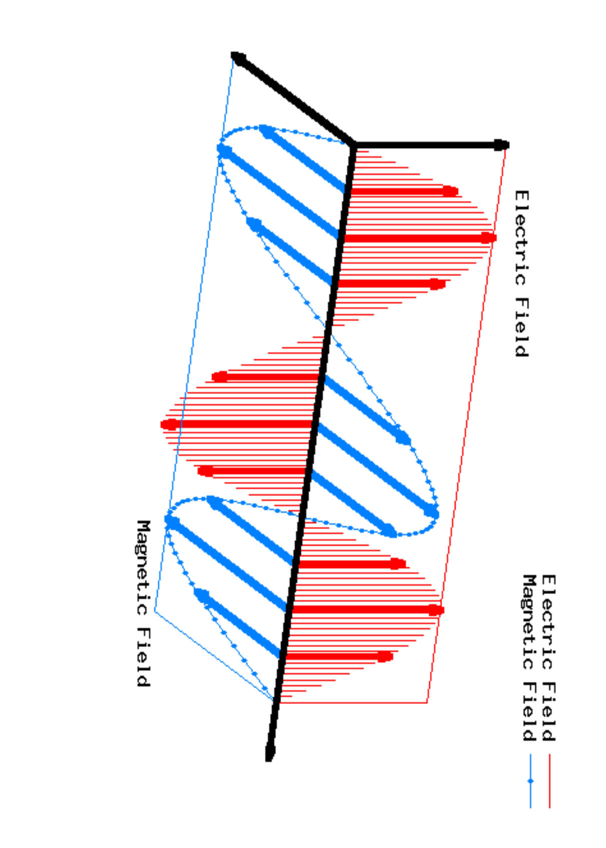
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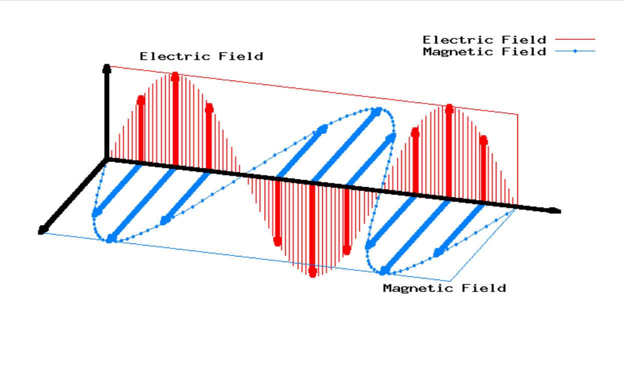
THANKS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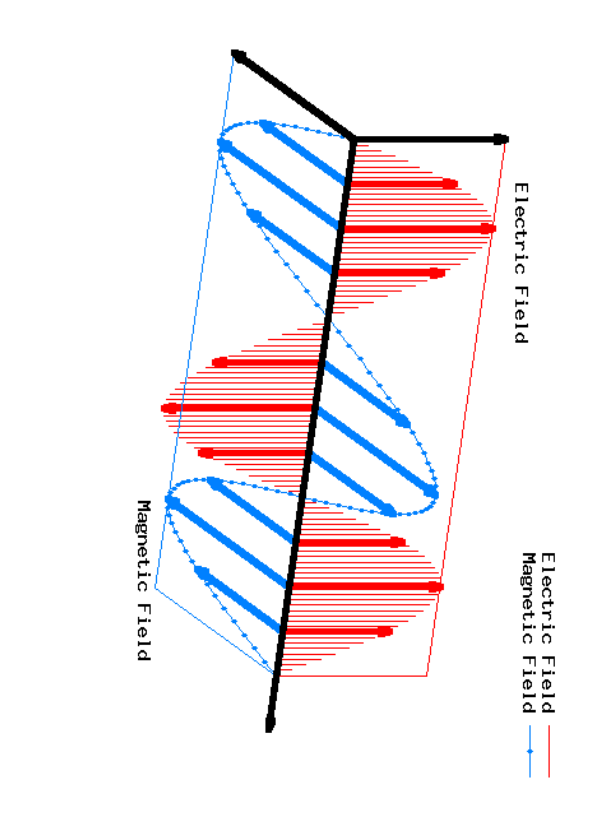
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