Subject Name-
ADVANCE QUANTUM MECHANICS

Subject Code- MPM-221

Teacher Name

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Syllabus??

MPM-221: ADVANCE QUANTUM MECHANICS **Credit 04 (3-1-0)**

Unit I: Formulation of Relativistic Quantum Theory

Relativistic Notations, The Klein-Gordon equation, Physical interpretation, Probability current density & Inadequacy of Klein-Gordon equation, Dirac relativistic equation & Mathematical formulation, α and β matrices and related algebra, Properties of four matrices α and β, Matrix representation of $\alpha_i^{\prime s}$ and β, True continuity equation and interpretation.

Unit II: Covariance of Dirac Equation

Covariant form of Dirac equation, Dirac gamma (γ) matrices, Representation and properties, Trace identities, fifth gamma matrix γ^5, Solution of Dirac equation for free particle (Plane wave solution), Dirac spinor, Helicity operator, Explicit form, Negative energy states

Unit III: Field Quantization

Introduction to quantum field theory, Lagrangian field theory, Euler–Lagrange equations, Hamiltonian formalism, Quantized Lagrangian field theory, Canonical commutation relations, The Klein-Gordon field, Second quantization, Hamiltonian and Momentum, Normal ordering, Fock space, The complex Klein-Gordan field: complex scalar field

Unit IV: Approximate Methods

Time independent perturbation theory, The Variational method, Estimation of ground state energy, The Wentzel-Kramers-Brillouin (WKB) method, Validity of the WKB approximation, Time-Dependent Perturbation theory, Transition probability, Fermi-Golden Rule

Books & References:

1: Advance Quantum Mechanics by J. J. Sakurai (Pearson Education India)

2: Relativistic Quantum Mechanics by James D. Bjorken and Sidney D. Drell (McGraw-Hill Book Company; New York, 1964).

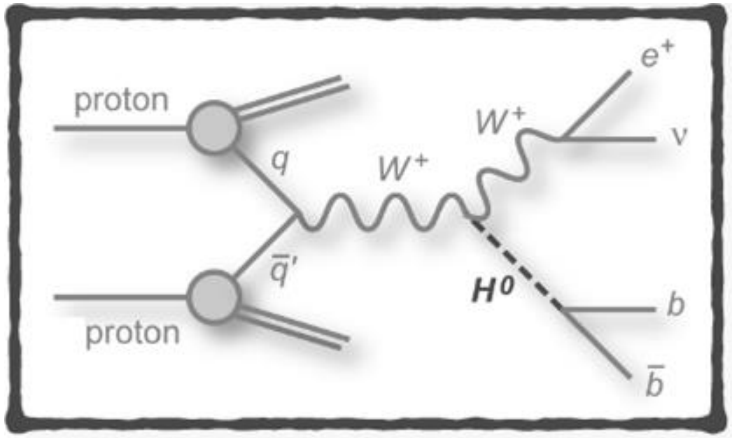
3: An Introduction to Relativistic Quantum Field Theory by S.S. Schweber (Harper & Row, New York, 1961).

4: Quantum Field Theory by F. Mandl & G. Shaw (John Wiley and Sons Ltd, 1984)

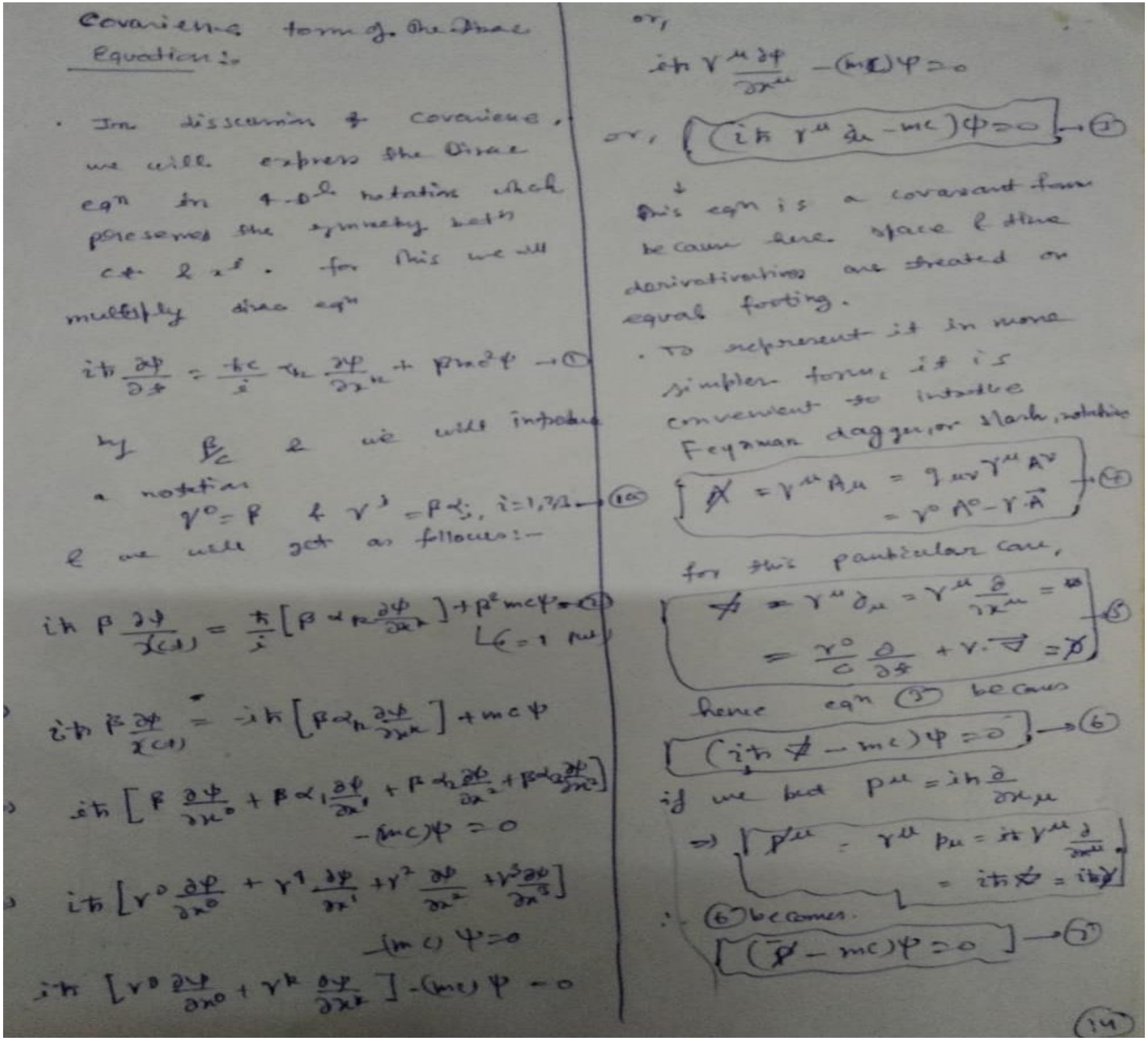
5: A First Book of Quantum Field Theory by A. Lahiri & P.B. Pal (Narosa Publishing House, New Delhi, 2000)

Session 2020-21

Lectures of Unit- II



Covariance form of the Dirac Equation



Covariance form of the Dirac Equation:-

- In discussion of covariance, we will express the Dirac eqn in 4-D notation which preserves the symmetry both ct & x^i. for this we will multiply dirac eqn

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\alpha_k\frac{\partial\psi}{\partial x^k} + \beta mc^2\psi \rightarrow ①$$

by β/c & we will introduce a notation

$$\gamma^0 = \beta \quad \& \quad \gamma^i = \beta\alpha_i,\ i=1,2,3 \rightarrow (1a)$$

& we will get as follows:-

$$i\hbar\beta\frac{\partial\psi}{\partial(ct)} = \frac{\hbar}{i}\left[\beta\alpha_k\frac{\partial\psi}{\partial x^k}\right] + \beta^2 mc\psi \rightarrow ②$$

$\lfloor(=1$ [illegible]

$$i\hbar\beta\frac{\partial\psi}{\partial(ct)} = -i\hbar\left[\beta\alpha_k\frac{\partial\psi}{\partial x^k}\right] + mc\psi$$

$$\Rightarrow i\hbar\left[\beta\frac{\partial\psi}{\partial x^0} + \beta\alpha_1\frac{\partial\psi}{\partial x^1} + \beta\alpha_2\frac{\partial\psi}{\partial x^2} + \beta\alpha_3\frac{\partial\psi}{\partial x^3}\right] - (mc)\psi = 0$$

$$\Rightarrow i\hbar\left[\gamma^0\frac{\partial\psi}{\partial x^0} + \gamma^1\frac{\partial\psi}{\partial x^1} + \gamma^2\frac{\partial\psi}{\partial x^2} + \gamma^3\frac{\partial\psi}{\partial x^3}\right] - (mc)\psi = 0$$

$$i\hbar\left[\gamma^0\frac{\partial\psi}{\partial x^0} + \gamma^k\frac{\partial\psi}{\partial x^k}\right] - (mc)\psi = 0$$

or,

$$i\hbar\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - (mc)\psi = 0$$

or,

$$\boxed{(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0} \rightarrow ③$$

↓

This eqn is a covariant form because here space & time derivatives are treated on equal footing.

- To represent it in more simpler form, it is convenient to introduce Feynman dagger, or slash, notation

$$\boxed{\begin{aligned}\not{A} &= \gamma^\mu A_\mu = g_{\mu\nu}\gamma^\mu A^\nu \\ &= \gamma^0 A^0 - \gamma\cdot\vec{A}\end{aligned}} \rightarrow ④$$

for this particular case,

$$\boxed{\begin{aligned}\not{\partial} &= \gamma^\mu\partial_\mu = \gamma^\mu\frac{\partial}{\partial x^\mu} = \\ &= \frac{\gamma^0}{c}\frac{\partial}{\partial t} + \gamma\cdot\vec{\nabla} = \not{\partial}\end{aligned}} \rightarrow ⑤$$

hence eqn ③ becomes

$$\boxed{(i\hbar\not{\partial} - mc)\psi = 0} \rightarrow ⑥$$

if we put $p^\mu = i\hbar\frac{\partial}{\partial x_\mu}$

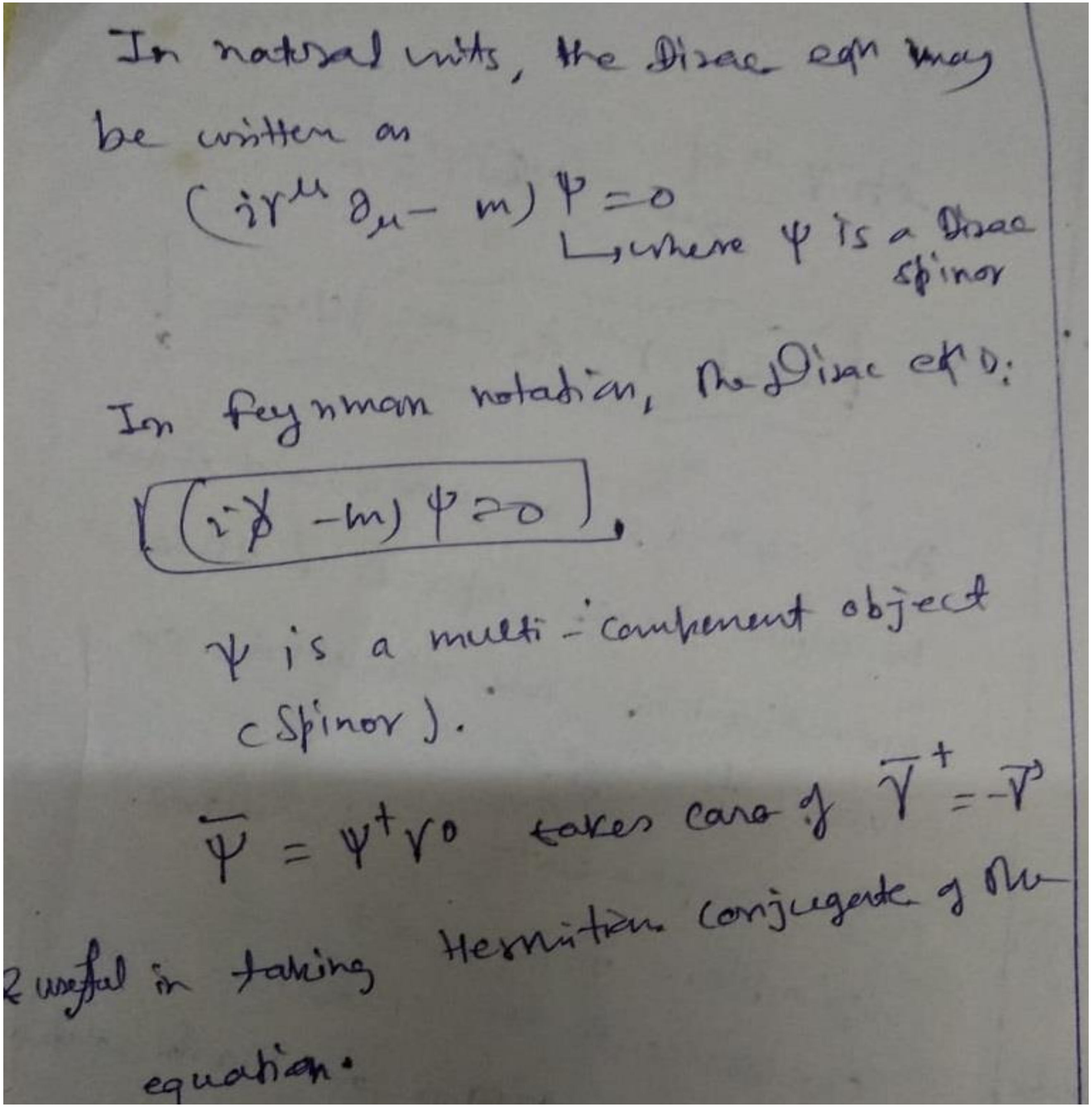
$$\Rightarrow \boxed{\begin{aligned}\not{p} &= \gamma^\mu p_\mu = i\hbar\gamma^\mu\frac{\partial}{\partial x^\mu} \\ &= i\hbar\not{\partial} = i\hbar\not{\partial}\end{aligned}}$$

∴ ⑥ becomes

$$\boxed{(\not{p} - mc)\psi = 0} \rightarrow ⑦$$

⑭

Covariance form of the Dirac Equation



In natural units, the Dirac eqn may be written as

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

↳ where ψ is a Dirac spinor

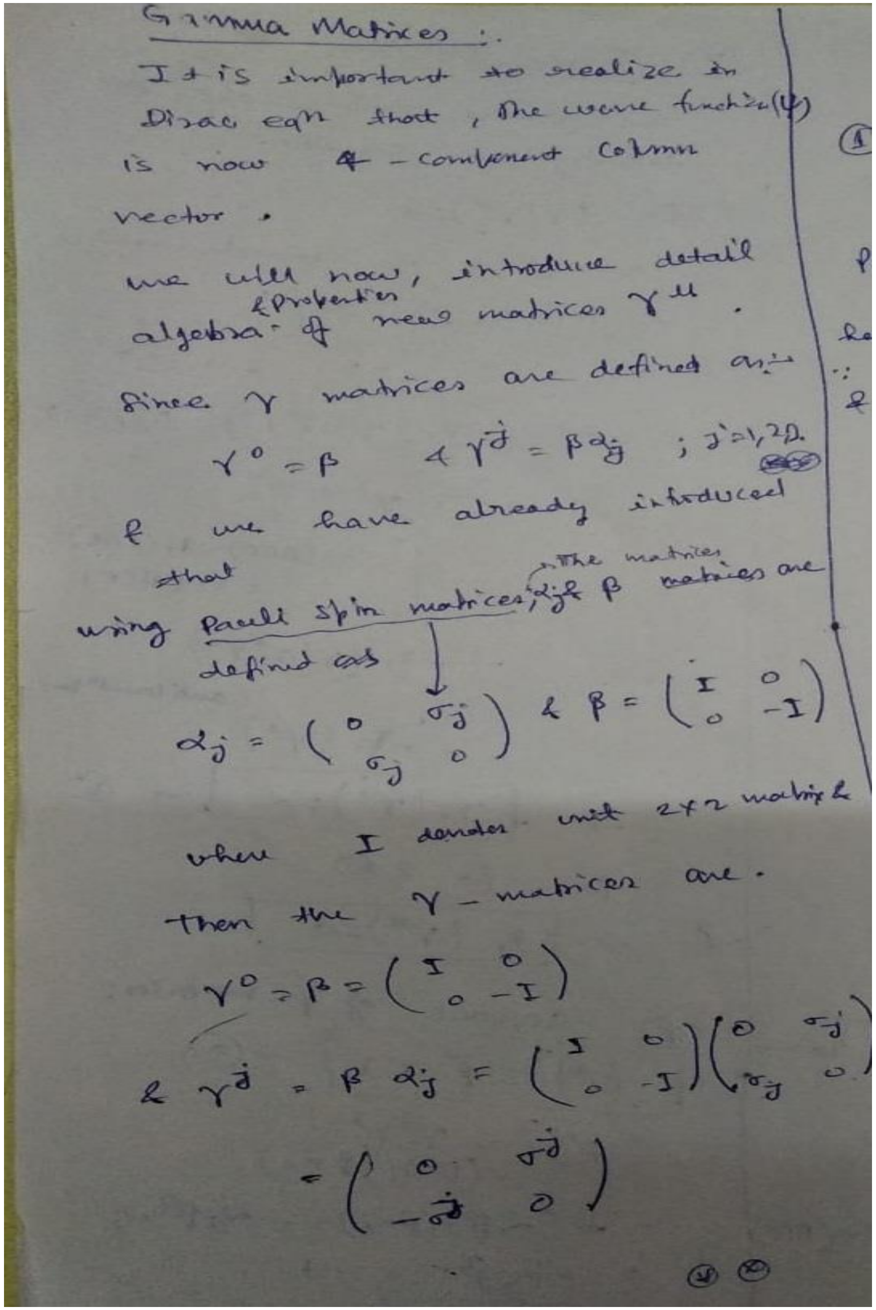
In Feynman notation, the Dirac eqn:

$$\boxed{(i\not{\partial} - m)\psi = 0}$$

ψ is a multi-component object (spinor).

$\bar{\psi} = \psi^{\dagger}\gamma^{0}$ takes care of $\overline{\gamma}^{\dagger} = -\overline{\gamma}$ & useful in taking Hermitian conjugate of the equation.

Gamma Matrices



Gamma Matrices :-

①

It is important to realize in Dirac eqn that, the wave function(ψ) is now 4-component column vector.

we will now, introduce detail algebra & properties of new matrices γ^{μ}.

Since γ matrices are defined as:-

$$\gamma^0 = \beta \quad \& \quad \gamma^j = \beta\alpha_j \quad ; \; j=1,2,3.$$

& we have already introduced that

using Pauli spin matrices, the matrices α_j & β matrices are defined as

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \& \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

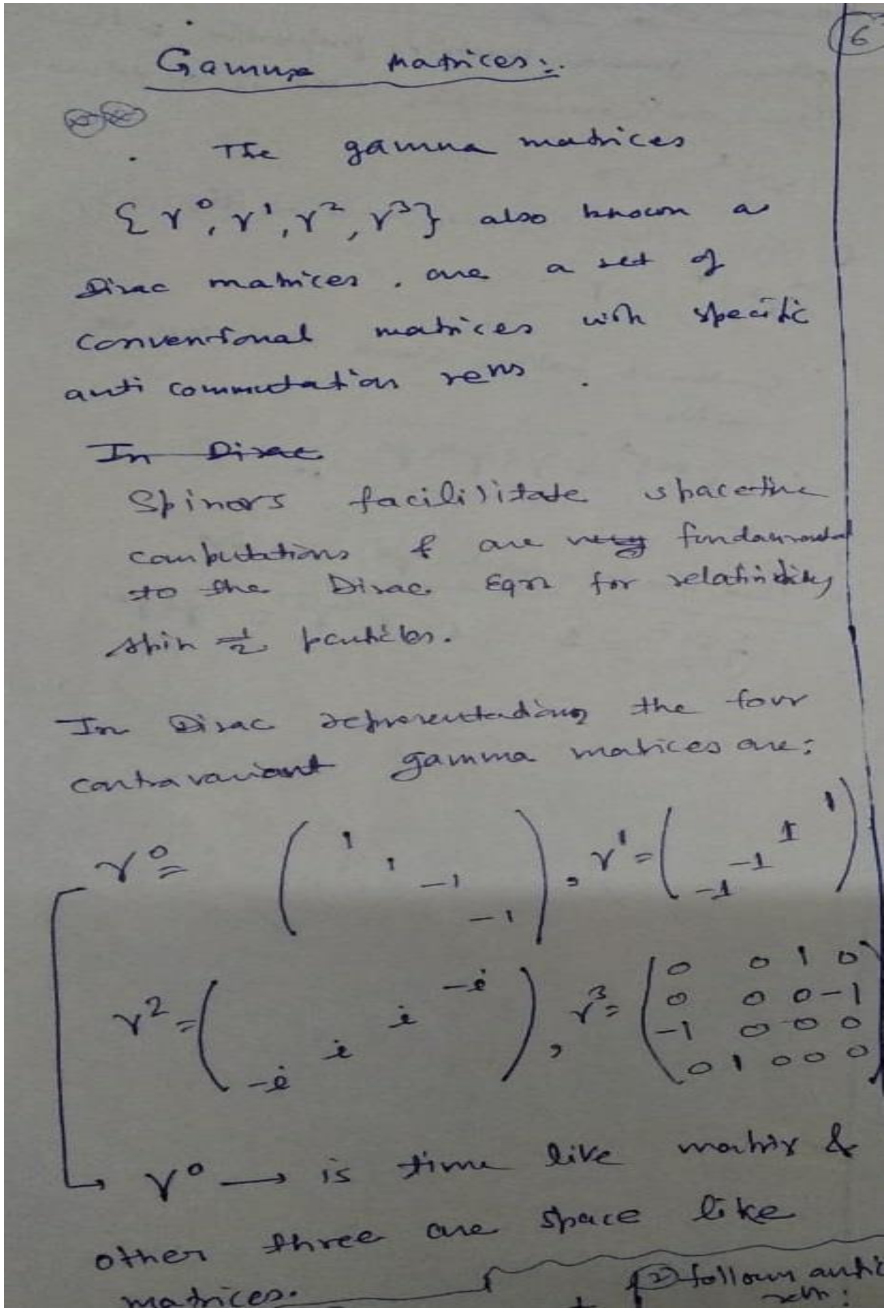
where I denotes unit 2×2 matrix &

Then the γ-matrices are.

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\& \quad \gamma^j = \beta\alpha_j = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$



⑥

Gamma matrices :-

. The gamma matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ also known as Dirac matrices, are a set of conventional matrices with specific anti commutation relns.

~~In Dirac~~

Spinors facilitate spacetime computations & are ~~very~~ fundamental to the Dirac Eqn for relativistic spin $\frac{1}{2}$ particles.

In Dirac representation the four contravariant gamma matrices are:

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

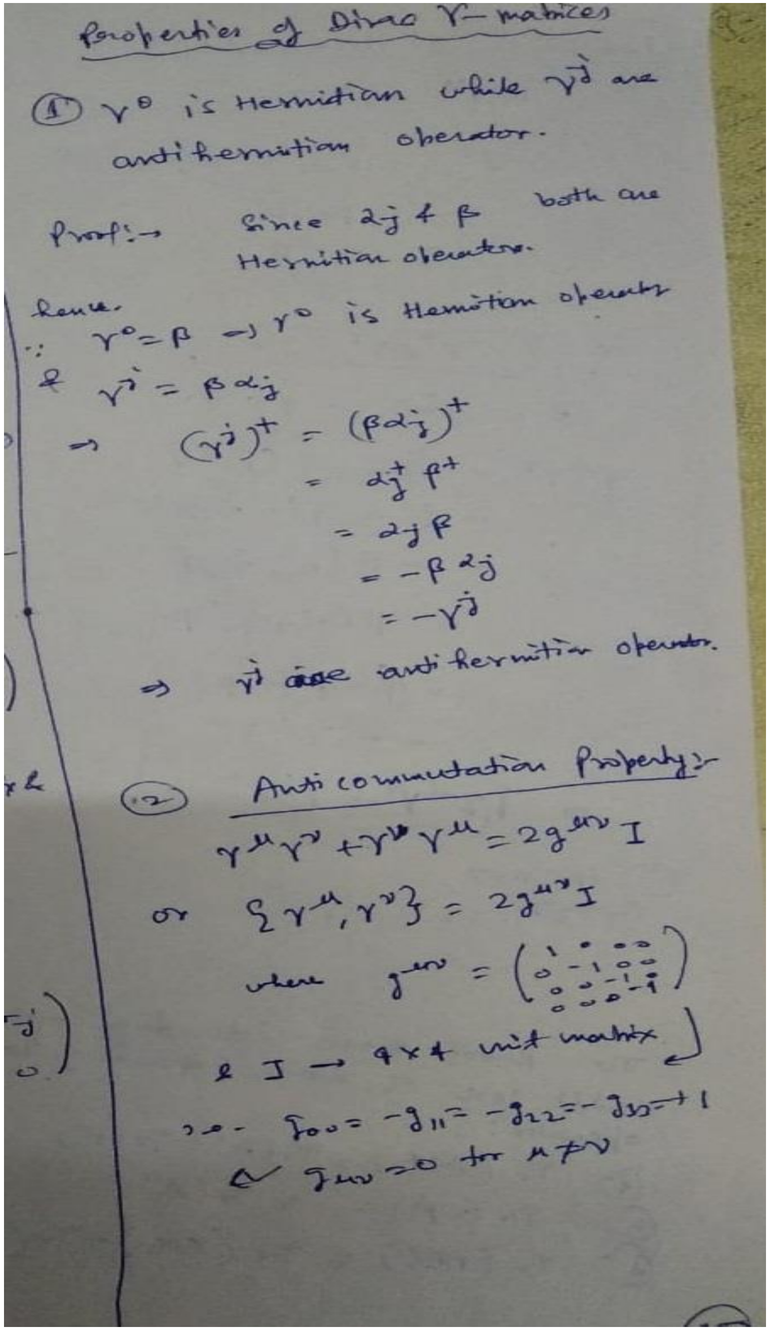
$$\gamma^2 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$\gamma^0 \longrightarrow$ is time like matrix & other three are space like matrices.

② follows anti

reln:

Properties of Gamma Matrices



Properties of Dirac γ– matrices

① γ^0 is Hermitian while γ^j are antihermitian operator.

Proof:→ Since α_j & β both are Hermitian operators.

hence,

$\because$ $\gamma^0=\beta \rightarrow \gamma^0$ is Hermitian operator

& $\gamma^j = \beta\alpha_j$

$$\Rightarrow (\gamma^j)^+ = (\beta\alpha_j)^+$$
$$= \alpha_j^+ \beta^+$$
$$= \alpha_j \beta$$
$$= -\beta\alpha_j$$
$$= -\gamma^j$$

$\Rightarrow$ γ^j are anti hermitian operator.

② Anti commutation Property:-

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I$$

or $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$

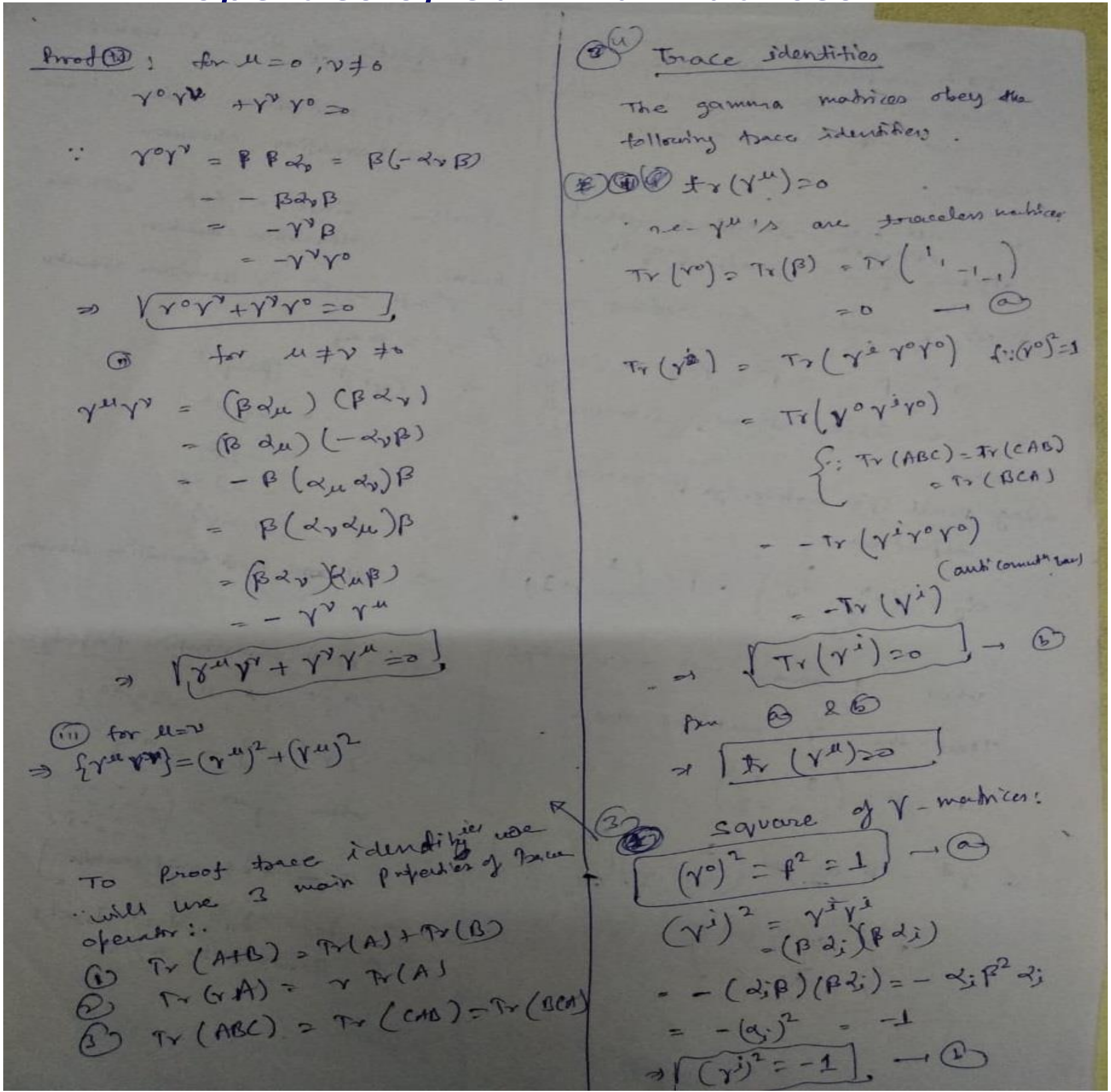
where $g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

& $I \rightarrow$ 4×4 unit matrix

i.e. $g_{00} = -g_{11} = -g_{22} = -g_{33} = +1$

& $g_{\mu\nu} = 0$ for $\mu \neq \nu$

Properties of Gamma Matrices



Proof (ii) : for $\mu = 0, \nu \neq 0$

$$\gamma^0\gamma^\nu + \gamma^\nu\gamma^0 = 0$$

$$\because \quad \gamma^0\gamma^\nu = \beta\beta\alpha_\nu = \beta(-\alpha_\nu\beta)$$

$$= -\beta\alpha_\nu\beta$$

$$= -\gamma^\nu\beta$$

$$= -\gamma^\nu\gamma^0$$

$$\Rightarrow \boxed{\gamma^0\gamma^\nu + \gamma^\nu\gamma^0 = 0}$$

(ii) for $\mu \neq \nu \neq 0$

$$\gamma^\mu\gamma^\nu = (\beta\alpha_\mu)(\beta\alpha_\nu)$$

$$= (\beta\alpha_\mu)(-\alpha_\nu\beta)$$

$$= -\beta(\alpha_\mu\alpha_\nu)\beta$$

$$= \beta(\alpha_\nu\alpha_\mu)\beta$$

$$= (\beta\alpha_\nu)(\alpha_\mu\beta)$$

$$= -\gamma^\nu\gamma^\mu$$

$$\Rightarrow \boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0}$$

(iii) for $\mu = \nu$

$$\Rightarrow \{\gamma^\mu\gamma^\mu\} = (\gamma^\mu)^2 + (\gamma^\mu)^2$$

To proof trace identities we will use 3 main properties of trace operator :

1. $Tr(A+B) = Tr(A) + Tr(B)$
2. $Tr(\gamma A) = \gamma\, Tr(A)$
3. $Tr(ABC) = Tr(CAB) = Tr(BCA)$

(2) Trace identities

The gamma matrices obey the following trace identities.

$$Tr(\gamma^\mu) = 0$$

i.e. γ^μ's are traceless matrices

$$Tr(\gamma^0) = Tr(\beta) = Tr\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}$$

$$= 0 \longrightarrow (a)$$

$$Tr(\gamma^i) = Tr(\gamma^i\gamma^0\gamma^0) \quad \{\because (\gamma^0)^2 = 1$$

$$= Tr(\gamma^0\gamma^i\gamma^0)$$

$$\{\because Tr(ABC) = Tr(CAB) = Tr(BCA)$$

$$= -Tr(\gamma^i\gamma^0\gamma^0)$$

(anti commutation law)

$$= -Tr(\gamma^i)$$

$$\Rightarrow \boxed{Tr(\gamma^i) = 0} \longrightarrow (b)$$

from (a) & (b)

$$\Rightarrow \boxed{Tr(\gamma^\mu) = 0}$$

(3) Square of γ-matrices:

$$\boxed{(\gamma^0)^2 = \beta^2 = 1} \longrightarrow (a)$$

$$(\gamma^i)^2 = \gamma^i\gamma^i$$

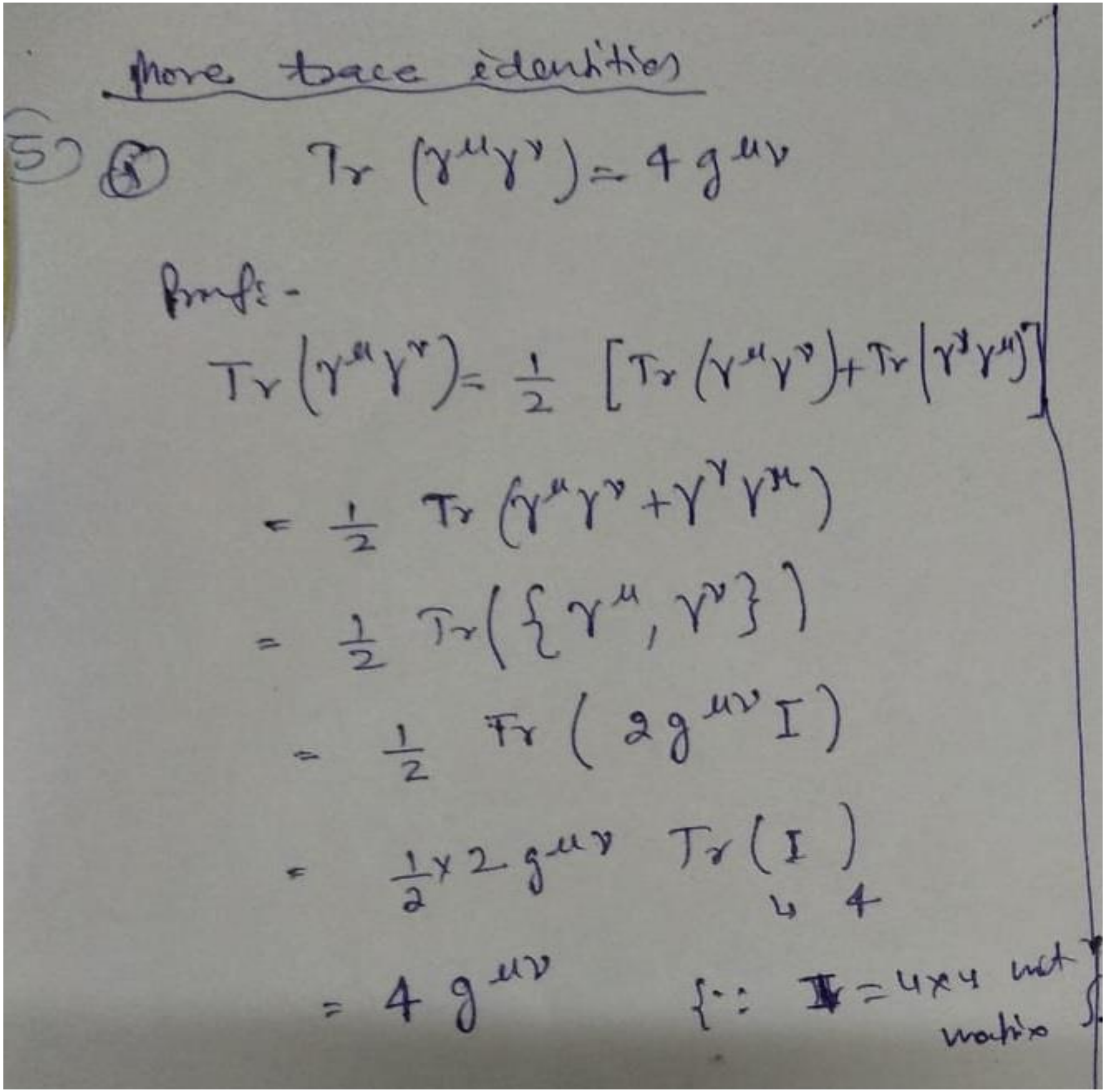
$$= (\beta\alpha_i)(\beta\alpha_i)$$

$$= -(\alpha_i\beta)(\beta\alpha_i) = -\alpha_i\beta^2\alpha_i$$

$$= -(\alpha_i)^2 = -1$$

$$\Rightarrow \boxed{(\gamma^i)^2 = -1} \longrightarrow (b)$$

Properties of Gamma Matrices



More trace identities

5) ⑤ $\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu}$

Proof:-

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) = \frac{1}{2}\left[\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) + \mathrm{Tr}(\gamma^{\nu}\gamma^{\mu})\right]$$

$$= \frac{1}{2}\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu})$$

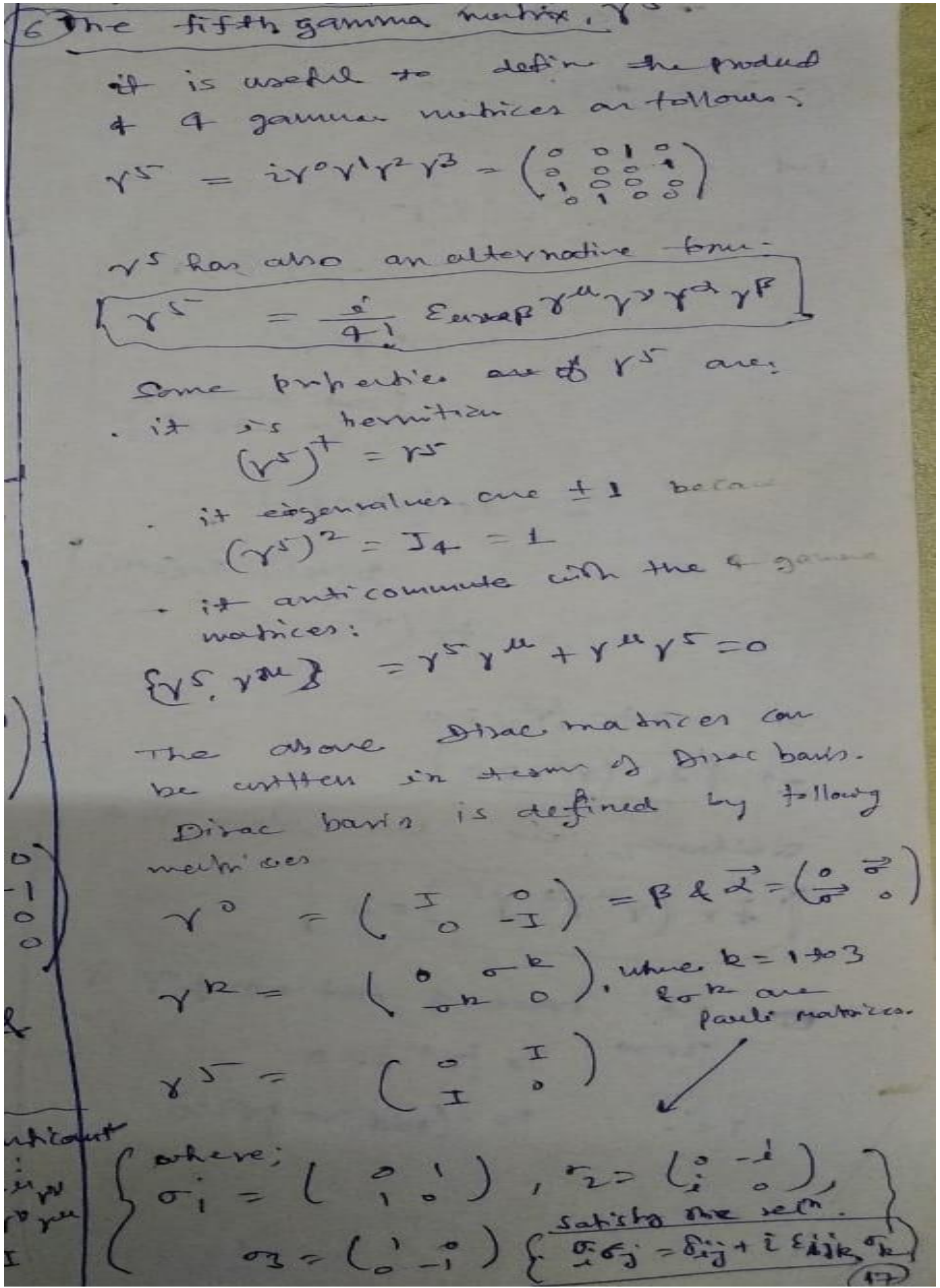
$$= \frac{1}{2}\mathrm{Tr}(\{\gamma^{\mu}, \gamma^{\nu}\})$$

$$= \frac{1}{2}\mathrm{Tr}(2g^{\mu\nu}I)$$

$$= \frac{1}{2}\times 2g^{\mu\nu}\,\underset{\hookrightarrow 4}{\mathrm{Tr}(I)}$$

$$= 4g^{\mu\nu}$$ $\{\because I = 4\times 4$ unit matrix$\}$

Properties of Gamma Matrices



⑥ The fifth gamma matrix, γ^5:

it is useful to defin the product of 4 gamma matrices as follows:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

γ^5 has also an alternative form:

$$\boxed{\gamma^5 = \frac{i}{4!}\varepsilon_{\mu\nu\alpha\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}}$$

Some properties are of γ^5 are:

- it is hermitian
  $$(\gamma^5)^{\dagger} = \gamma^5$$
- it eigenvalues are ± 1 becau
  $$(\gamma^5)^2 = I_4 = 1$$
- it anticommute with the 4 gamma matrices:
  $$\{\gamma^5, \gamma^{\mu}\} = \gamma^5\gamma^{\mu} + \gamma^{\mu}\gamma^5 = 0$$

The above Dirac matrices can be written in term of Dirac basis. Dirac basis is defined by follow matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \beta \;\&\; \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$ where $k = 1$ to 3 & σ^k are Pauli matrices.

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where;

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

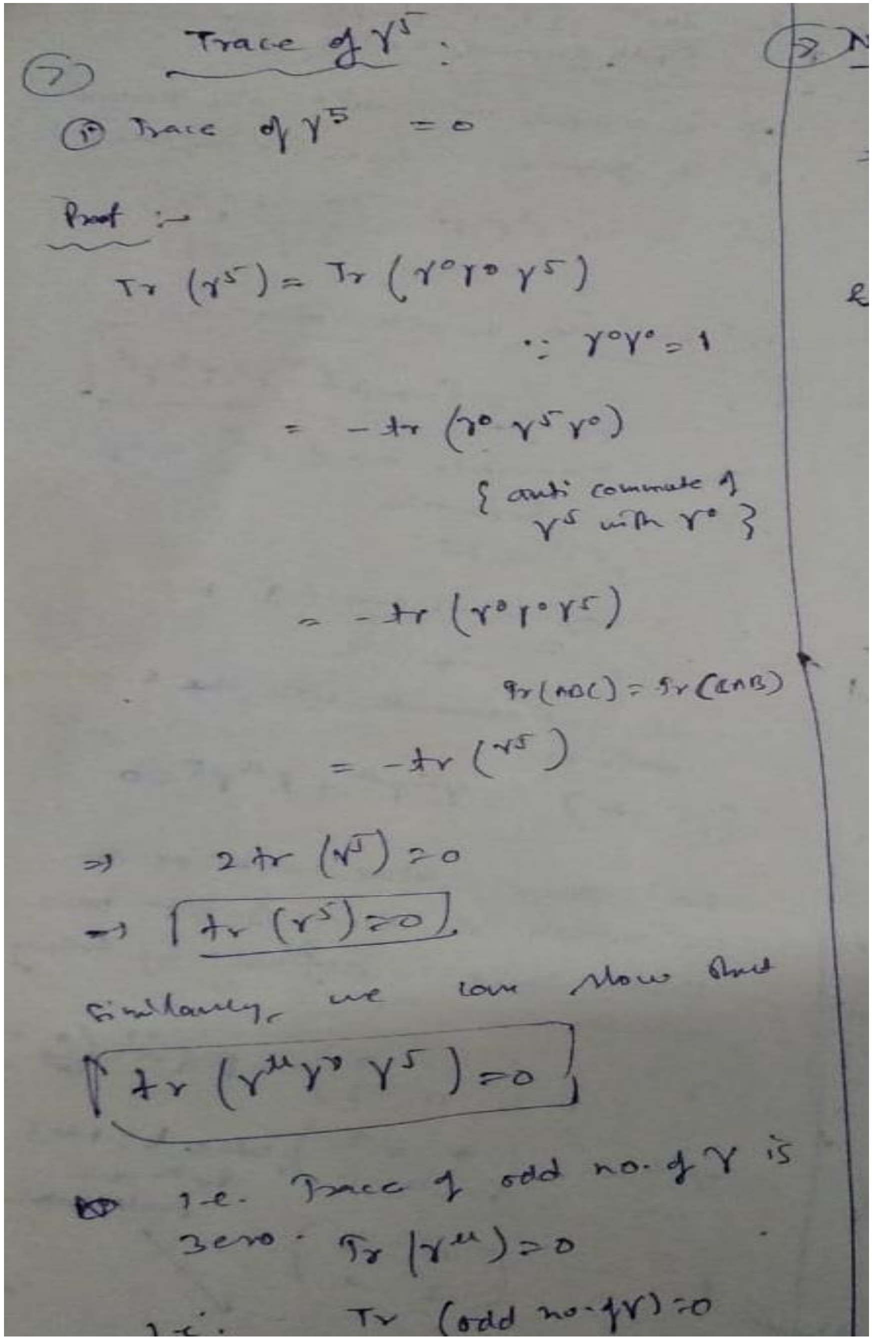
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the relⁿ.

$$\sigma_i\sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\,\sigma_k$$

(17)

Properties of Gamma Matrices



⑦ Trace of γ^5 :

(i) Trace of $\gamma^5 = 0$

Proof :-

$$Tr(\gamma^5) = Tr(\gamma^0\gamma^0\gamma^5)$$

$\because \gamma^0\gamma^0 = 1$

$$= -tr(\gamma^0\gamma^5\gamma^0)$$

{anti commute of γ^5 with γ^0 }

$$= -tr(\gamma^0\gamma^0\gamma^5)$$

$tr(ABC) = tr(CAB)$

$$= -tr(\gamma^5)$$

$$\Rightarrow 2\,tr(\gamma^5) = 0$$

$$\Rightarrow \boxed{tr(\gamma^5) = 0}$$

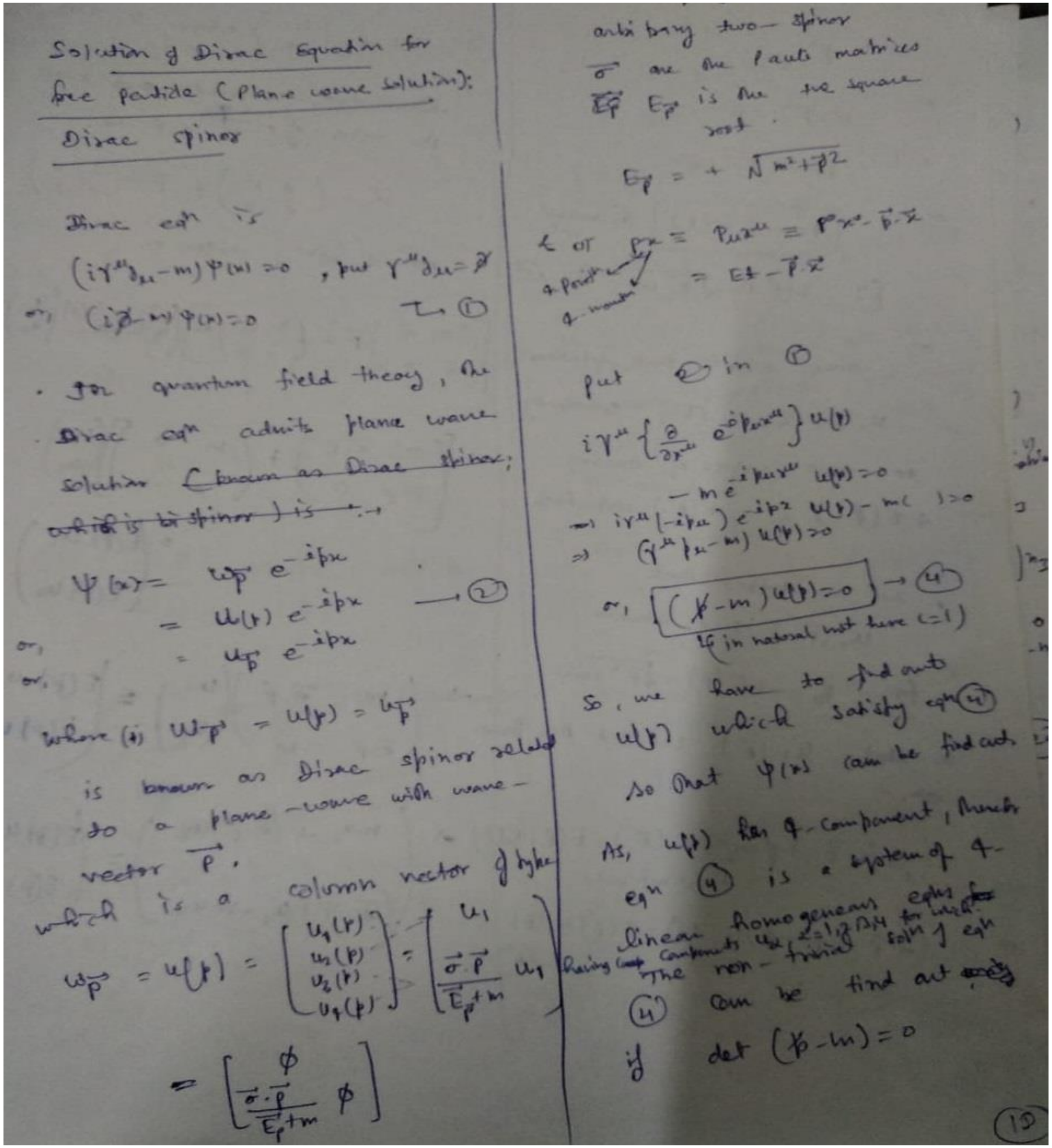
Similarly, we can show that

$$\boxed{tr(\gamma^\mu\gamma^\nu\gamma^5) = 0}$$

i.e. Trace of odd no. of γ is zero. $Tr(\gamma^\mu) = 0$

i.e. $Tr(\text{odd no. of } \gamma) = 0$

Solution of Dirac Equation for free particles: Plane wave solution



Solution of Dirac Equation for free particle (Plane wave solution):

Dirac spinor

Dirac eqn is

$(i\gamma^{\mu}\partial_{\mu}-m)\psi(x)=0$, put $\gamma^{\mu}\partial_{\mu}=\not\partial$

or, $(i\not\partial-m)\psi(x)=0$ $\to$ ①

- For quantum field theory, the Dirac eqn admits plane wave solution ~~(known as Dirac spinor; which is bi spinor) is~~ :→

$$\psi(x)= w_{\vec{p}}\, e^{-ipx} = u(p)\, e^{-ipx} \longrightarrow ②$$

or, $= u_{\vec{p}}\, e^{-ipx}$

or,

where (i) $w_{\vec{p}} = u(p) = u_{\vec{p}}$ is known as Dirac spinor related to a plane-wave with wave-vector $\vec{p}$.

which is a column vector of type

$$w_{\vec{p}} = u(p) = \begin{bmatrix} u_1(p)\\ u_2(p)\\ u_3(p)\\ u_4(p)\end{bmatrix} = \begin{bmatrix} u_1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,u_1 \end{bmatrix}$$

$$= \begin{bmatrix} \phi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\phi \end{bmatrix}$$

arbitrary two-spinor

$\vec{\sigma}$ are the Pauli matrices

~~E_p~~ E_p is the +ve square root.

$$E_p = +\sqrt{m^2+p^2}$$

& or $px \equiv p_\mu x^\mu \equiv p^0x^0-\vec{p}\cdot\vec{x}$

$= Et-\vec{p}\cdot\vec{x}$

4-position

4-momentum

put ② in ①

$$i\gamma^{\mu}\left\{\frac{\partial}{\partial x^{\mu}}\, e^{-ip_\mu x^\mu}\right\}u(p) - m\, e^{-ip_\mu x^\mu}\, u(p)=0$$

$\Rightarrow i\gamma^{\mu}(-ip_\mu)\, e^{-ipx}\, u(p) - m(\quad)=0$

$\Rightarrow (\gamma^{\mu}p_{\mu}-m)\,u(p)=0$

or, $\boxed{(\not p-m)\,u(p)=0} \to ④$

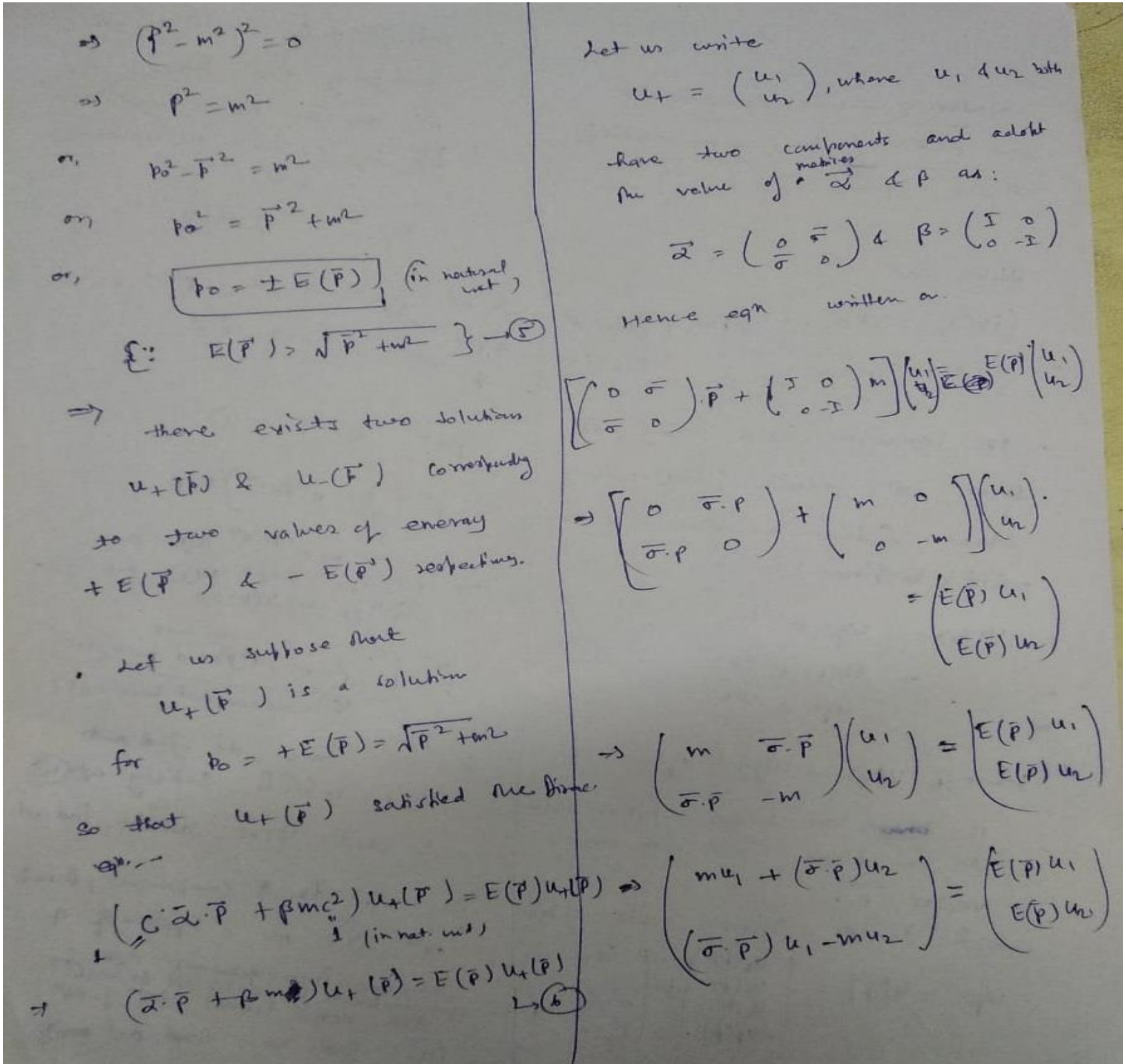
(in natural unit here c=1)

So, we have to find out u(p) which satisfy eqn ④ so that ψ(x) can be find out.

As, u(p) has 4-component, hence eqn ④ is a system of 4-linear homogeneous eqns for which having components u_α, $\alpha=1,2,3,4$. The non-trivial soln of eqn ④ can be find out if $\det(\not p-m)=0$

⑩

Solution of Dirac Equation for free particles: Continue…



$\Rightarrow (p^2 - m^2)^2 = 0$

$\Rightarrow p^2 = m^2$

or, $p_0^2 - \vec{p}^{\,2} = m^2$

or, $p_0^2 = \vec{p}^{\,2} + m^2$

or, $\boxed{p_0 = \pm E(\bar{p})}$ (in natural unit)

$\left\{\because E(\vec{p}) = \sqrt{\bar{p}^2 + m^2}\right\} \rightarrow$ ⑤

$\Rightarrow$ there exists two solutions $u_+(\bar{p})$ & $u_-(\vec{p})$ corresponding to two values of energy $+E(\vec{p})$ & $-E(\vec{p})$ respectively.

- Let us suppose that $u_+(\vec{p})$ is a solution for $p_0 = +E(\bar{p}) = \sqrt{\bar{p}^2 + m^2}$ so that $u_+(\vec{p})$ satisfied the Dirac eqn.-

$$(\underset{1}{c}\,\bar{\alpha}\cdot\vec{p} + \beta m\underset{1}{c^2})\,u_+(\vec{p}) = E(\vec{p})\,u_+(\vec{p})$$

(in nat. unit)

$$\Rightarrow (\bar{\alpha}\cdot\bar{p} + \beta m)\,u_+(\bar{p}) = E(\bar{p})\,u_+(\bar{p})$$ ⑥

Let us write

$u_+ = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where u_1 & u_2 both have two components and adopt the value of matrices $\vec{\alpha}$ & β as:

$$\vec{\alpha} = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} \;\&\; \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Hence eqn written as

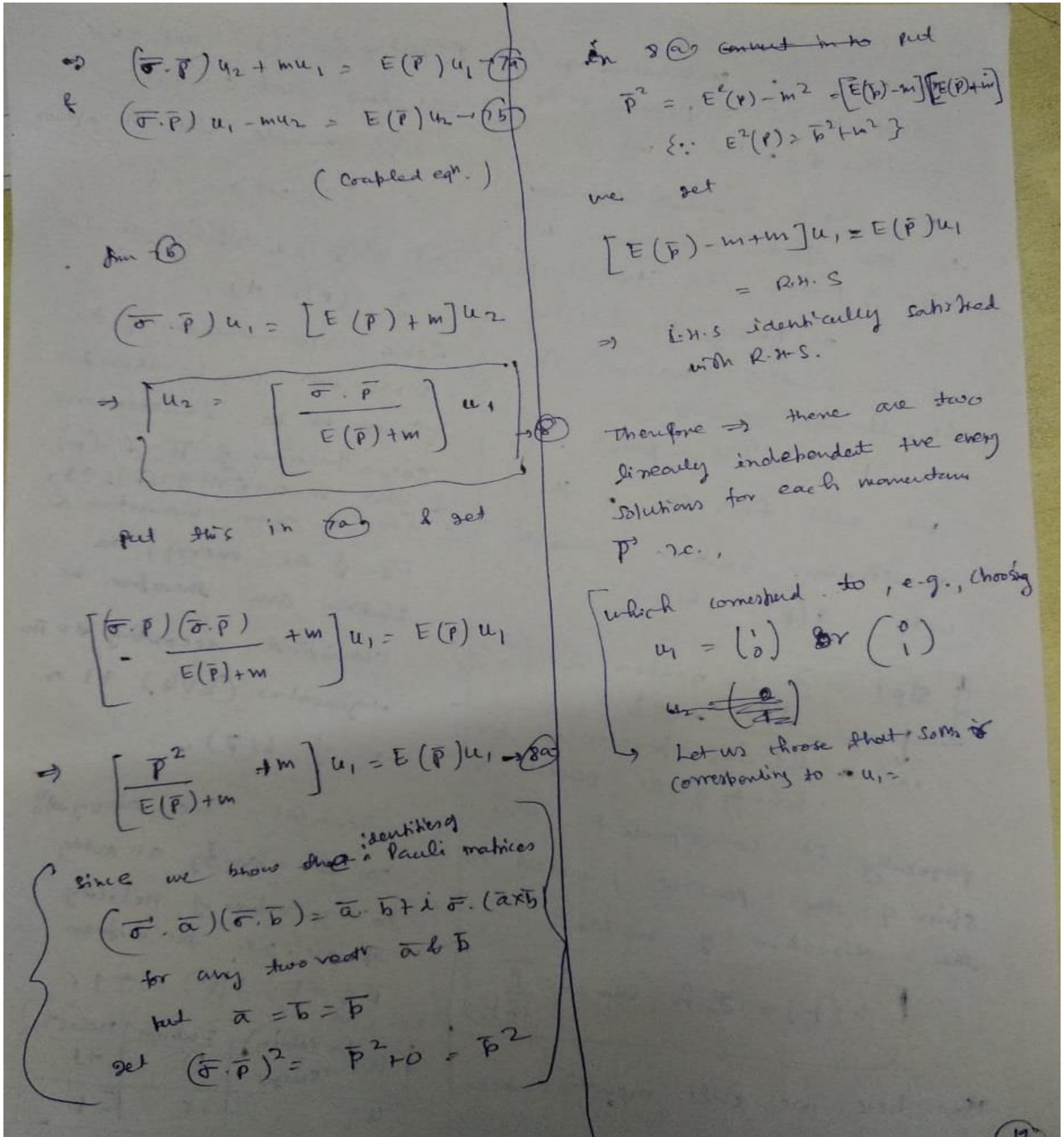
$$\left[\begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix}\cdot\vec{p} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} m\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = E(\vec{p})\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Rightarrow \left[\begin{pmatrix} 0 & \bar{\sigma}\cdot p \\ \bar{\sigma}\cdot p & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} E(\bar{p})\,u_1 \\ E(\bar{p})\,u_2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} m & \bar{\sigma}\cdot\vec{p} \\ \bar{\sigma}\cdot\bar{p} & -m \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} E(\bar{p})\,u_1 \\ E(\bar{p})\,u_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} mu_1 + (\bar{\sigma}\cdot\bar{p})u_2 \\ (\bar{\sigma}\cdot\vec{p})\,u_1 - mu_2 \end{pmatrix} = \begin{pmatrix} E(\bar{p})\,u_1 \\ E(\bar{p})\,u_2 \end{pmatrix}$$

Solution of Dirac Equation for free particles: Continue



$$\Rightarrow \quad (\bar{\sigma}\cdot\bar{p})u_2 + mu_1 = E(\bar{p})u_1 \rightarrow ⑦$$

$$\& \quad (\bar{\sigma}\cdot\bar{p})u_1 - mu_2 = E(\bar{p})u_2 \rightarrow ⑦b$$

(Coupled eqn.)

from ⑦b

$$(\bar{\sigma}\cdot\bar{p})u_1 = [E(\bar{p})+m]u_2$$

$$\Rightarrow \boxed{u_2 = \left[\frac{\bar{\sigma}\cdot\bar{p}}{E(\bar{p})+m}\right]u_1} \rightarrow ⑧$$

put this in ⑦a & get

$$\left[\frac{(\bar{\sigma}\cdot\bar{p})(\bar{\sigma}\cdot\bar{p})}{E(\bar{p})+m} + m\right]u_1 = E(\bar{p})u_1$$

$$\Rightarrow \left[\frac{\bar{p}^2}{E(\bar{p})+m} + m\right]u_1 = E(\bar{p})u_1 \rightarrow ⑧a$$

Since we know ~~that~~ identities of Pauli matrices

$$(\vec{\sigma}\cdot\bar{a})(\bar{\sigma}\cdot\bar{b}) = \bar{a}\cdot\bar{b} + i\,\bar{\sigma}\cdot(\bar{a}\times\bar{b})$$

for any two vectr $\bar{a}$ & $\bar{b}$

put $\bar{a} = \bar{b} = \bar{p}$

get $(\bar{\sigma}\cdot\bar{p})^2 = \bar{p}^2 + 0 = \bar{p}^2$

In 8ⓐ ~~convert in to~~ put

$$\bar{p}^2 = E^2(p) - m^2 = [E(p)-m][E(p)+m]$$

$$\{\because E^2(p) = \bar{p}^2 + m^2\}$$

we get

$$[E(\bar{p}) - m + m]u_1 = E(\bar{p})u_1$$

$$= \text{R.H.S}$$

$\Rightarrow$ L.H.S identically satisfied with R.H.S.

Therefore $\Rightarrow$ there are two linearly independent +ve energy solutions for each momentum $\vec{p}$ i.e.,

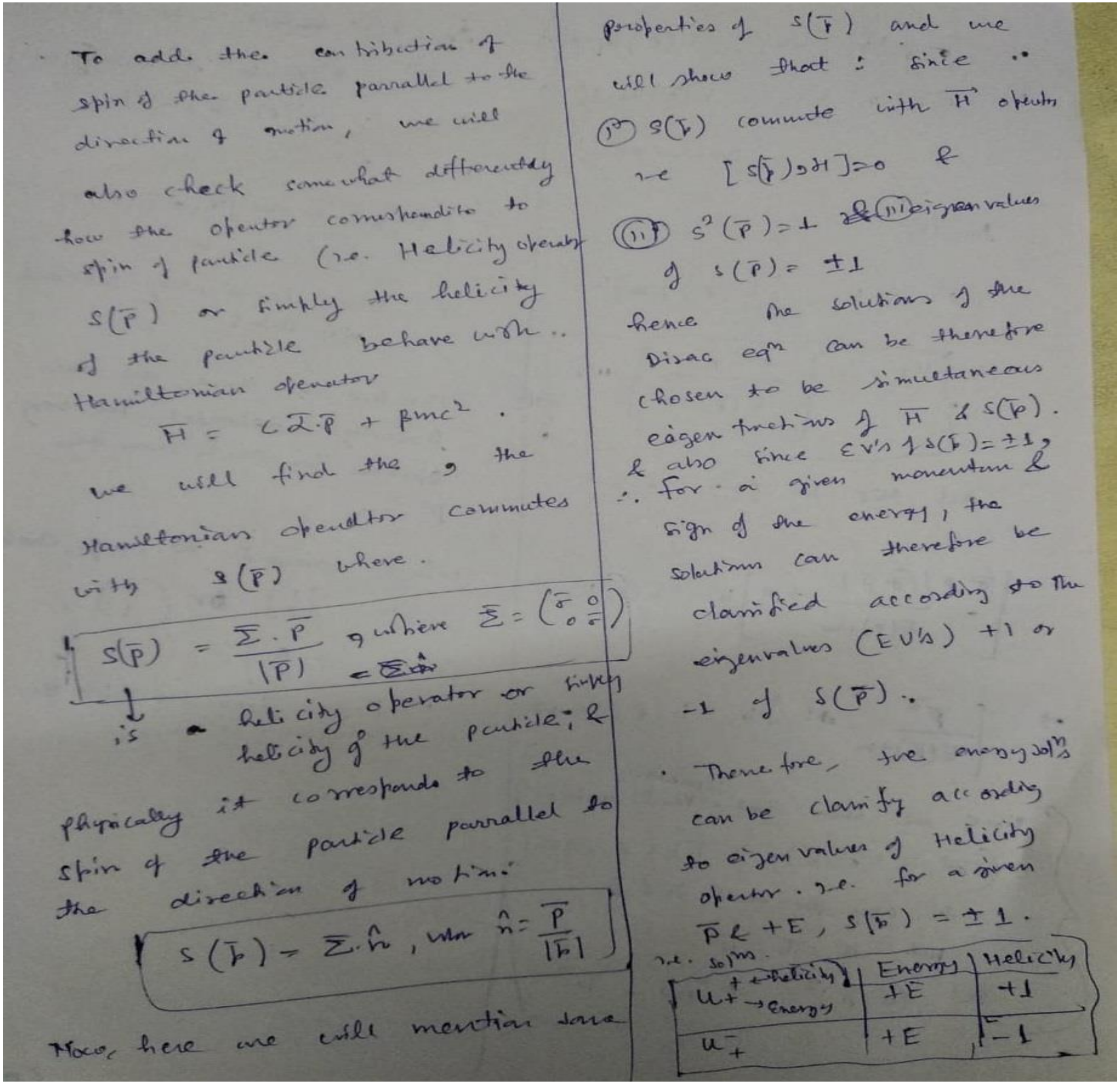
which correspond to, e.g., choosing

$$u_1 = \begin{pmatrix}1\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\1\end{pmatrix}$$

~~$u_2 = \left(\frac{0}{4}\right)$~~

$\rightarrow$ Let us choose that soln of corresponding to $u_1 =$

Solution of Dirac Equation for free particles: Continue



- To add the contribution of spin of the particle parrallel to the direction of motion, we will also check somewhat differently how the operator correspondito to spin of particle (i.e. Helicity operator $S(\overline{P})$ or simply the helicity of the particle behave with Hamiltonian operator

$$\overline{H} = c\overline{\alpha}\cdot\overline{p} + \beta mc^2 .$$

we will find the, the Hamiltonian operator commutes with $S(\overline{P})$ where.

$$\boxed{S(\overline{P}) = \frac{\overline{\Sigma}\cdot\overline{P}}{|\overline{P}|} , \text{ where } \overline{\Sigma} = \begin{pmatrix} \overline{\sigma} & 0 \\ 0 & \overline{\sigma} \end{pmatrix} = \overline{\Sigma}\cdot\hat{n}}$$

is a helicity operator or simply helicity of the particle; & physically it corresponds to the spin of the particle parrallel to the direction of motion.

$$\boxed{S(\overline{P}) = \overline{\Sigma}\cdot\hat{n} , \text{ where } \hat{n} = \frac{\overline{P}}{|\overline{P}|}}$$

Now, here we will mention some properties of $S(\overline{P})$ and we will show that: since

(i) $S(\overline{P})$ commute with $\overline{H}$ operator

i.e $[S(\overline{P}), H] = 0$ &

(ii) $S^2(\overline{P}) = 1$ & (iii) eigen values of $S(\overline{P}) = \pm 1$

hence the solutions of the Dirac eq^n can be therefore chosen to be simultaneous eigen functions of $\overline{H}$ & $S(\overline{P})$.

& also since EV's of $S(\overline{P}) = \pm 1$,

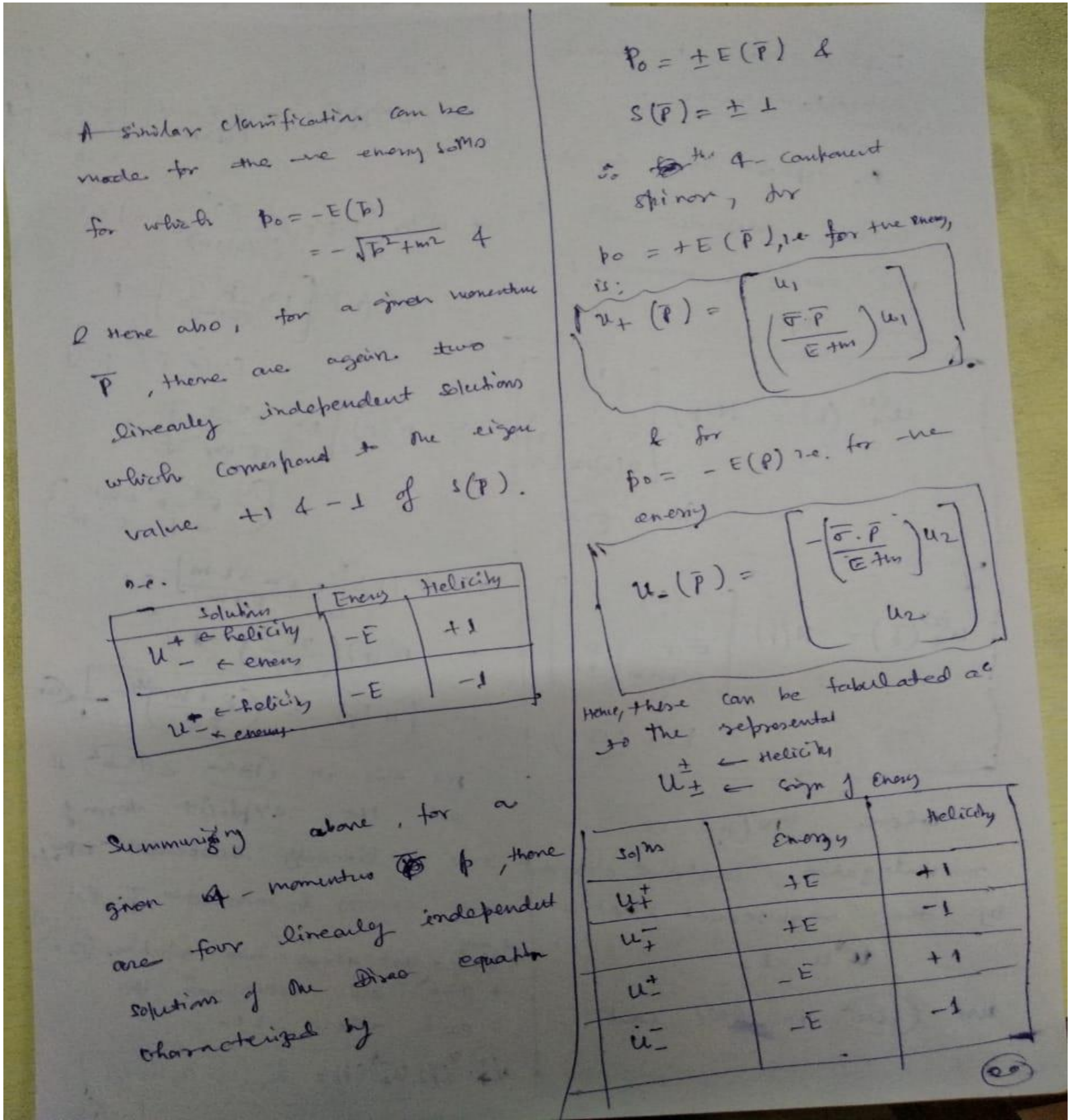
∴ for a given momentum & sign of the energy, the solutions can therefore be classified according to the eigenvalues (EV's) +1 or -1 of $S(\overline{P})$..

- Therefore, +ve energy sol^{n}s can be classify according to eigen values of Helicity operator. i.e. for a given $\overline{P}$ & $+E$, $S(\overline{P}) = \pm 1$.

i.e. sol^{n}s.

	Energy	Helicity
u_{+}^{+} + → helicity + → Energy	$+E$	$+1$
u_{+}^{-}	$+E$	-1

Solution of Dirac Equation for free particles: Continue



A similar classification can be made for the -ve energy solns for which $p_0 = -E(p)$

$= -\sqrt{p^2+m^2}$ &

& Here also, for a given momentum $\bar{p}$, there are again two linearly independent solutions which correspond to the eigen value +1 & -1 of $s(p)$.

i.e.

Solution	Energy	Helicity
u^{+}_{-} ← helicity ← energy	−E	+1
u^{+}_{-} ← helicity ← energy	−E	−1

Summarizing above, for a given 4-momentum ~~p~~ p, there are four linearly independent solutions of the Dirac equation characterized by

$P_0 = \pm E(\bar{p})$ &

$s(\bar{p}) = \pm 1$

∴ the 4-component spinor, for $p_0 = +E(\bar{P})$, i.e. for +ve energy, is:

$$\boxed{u_+(\bar{p}) = \begin{bmatrix} u_1 \\ \left(\frac{\bar{\sigma}\cdot\bar{p}}{E+m}\right)u_1 \end{bmatrix}}.$$

& for $p_0 = -E(p)$ i.e. for -ve energy

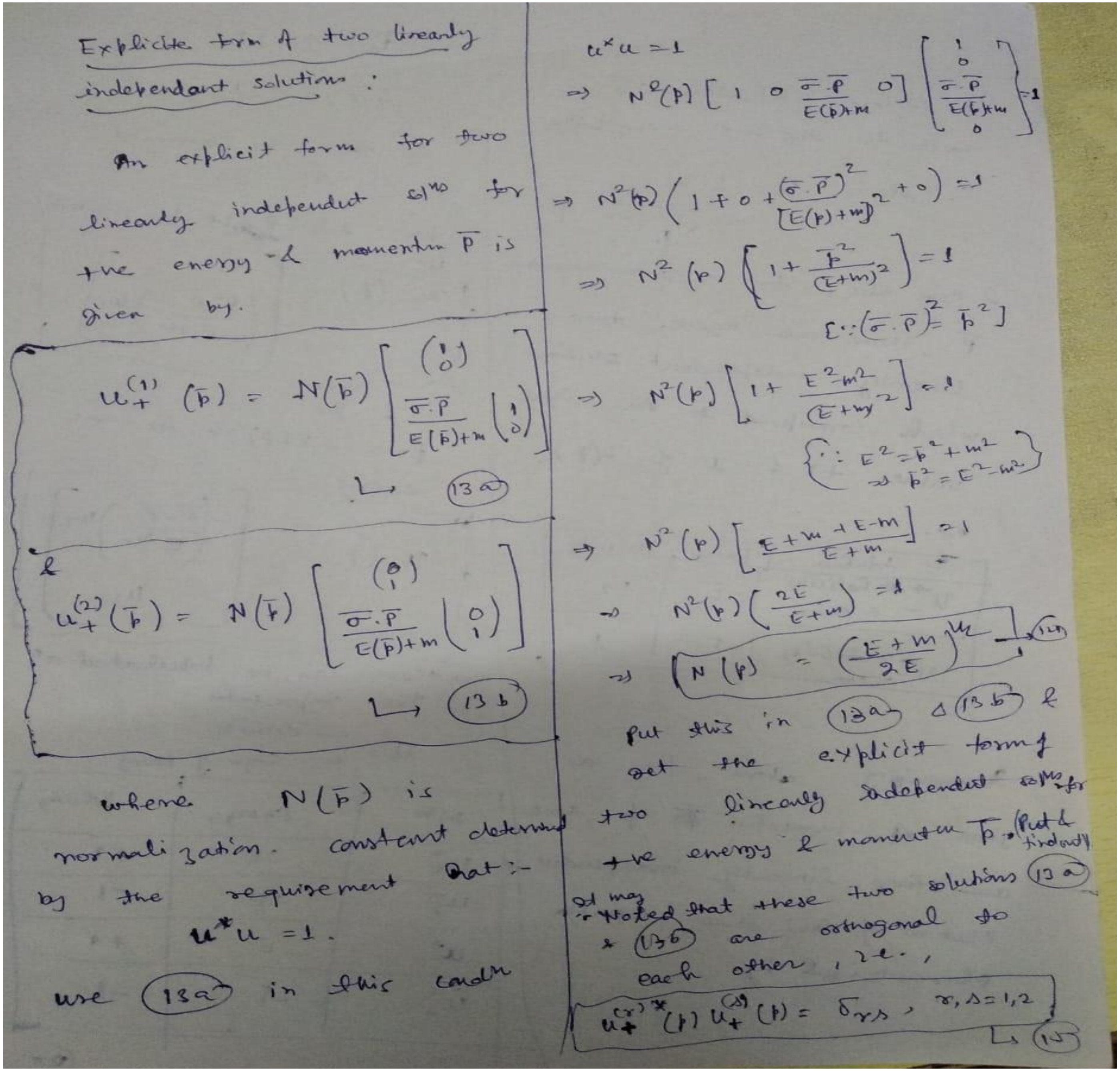
$$\boxed{u_-(\bar{p}) = \begin{bmatrix} -\left(\frac{\bar{\sigma}\cdot\bar{p}}{E+m}\right)u_2 \\ u_2 \end{bmatrix}}.$$

Hence, these can be tabulated as to the represental

$u^{\pm}_{\pm}$ ← Helicity ← sign of Energy

Solns	Energy	Helicity
u^{+}_{+}	+E	+1
u^{-}_{+}	+E	−1
u^{+}_{-}	−E	+1
u^{-}_{-}	−E	−1

Solution of Dirac Equation for free particles: Continue



Explicite form of two linearly independant solutions :

An explicit form for two linearly independent solⁿˢ for +ve energy & momentum $\bar{p}$ is given by.

$$u_+^{(1)}(\bar{p}) = N(\bar{p}) \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\bar{\sigma}.\bar{p}}{E(\bar{p})+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} \rightarrow (13a)$$

&

$$u_+^{(2)}(\bar{p}) = N(\bar{p}) \begin{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\bar{\sigma}.\bar{p}}{E(\bar{p})+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \rightarrow (13b)$$

where $N(\bar{p})$ is normalization constant determined by the requirement that :-

$$u^* u = 1 .$$

use (13a) in this condⁿ

$$u^* u = 1$$

$$\Rightarrow N^2(p) \left[1 \quad 0 \quad \frac{\bar{\sigma}.\bar{p}}{E(\bar{p})+m} \quad 0 \right] \begin{bmatrix} 1 \\ 0 \\ \frac{\bar{\sigma}.\bar{p}}{E(\bar{p})+m} \\ 0 \end{bmatrix} = 1$$

$$\Rightarrow N^2(p) \left(1 + 0 + \frac{(\bar{\sigma}.\bar{p})^2}{[E(p)+m]^2} + 0 \right) = 1$$

$$\Rightarrow N^2(p) \left[1 + \frac{\bar{p}^2}{(E+m)^2} \right] = 1$$

$$[\because (\bar{\sigma}.\bar{p})^2 = \bar{p}^2]$$

$$\Rightarrow N^2(p) \left[1 + \frac{E^2 - m^2}{(E+m)^2} \right] = 1$$

$$\left\{ \because E^2 = \bar{p}^2 + m^2 \Rightarrow \bar{p}^2 = E^2 - m^2 \right\}$$

$$\Rightarrow N^2(p) \left[\frac{E+m+E-m}{E+m} \right] = 1$$

$$\Rightarrow N^2(p) \left(\frac{2E}{E+m} \right) = 1$$

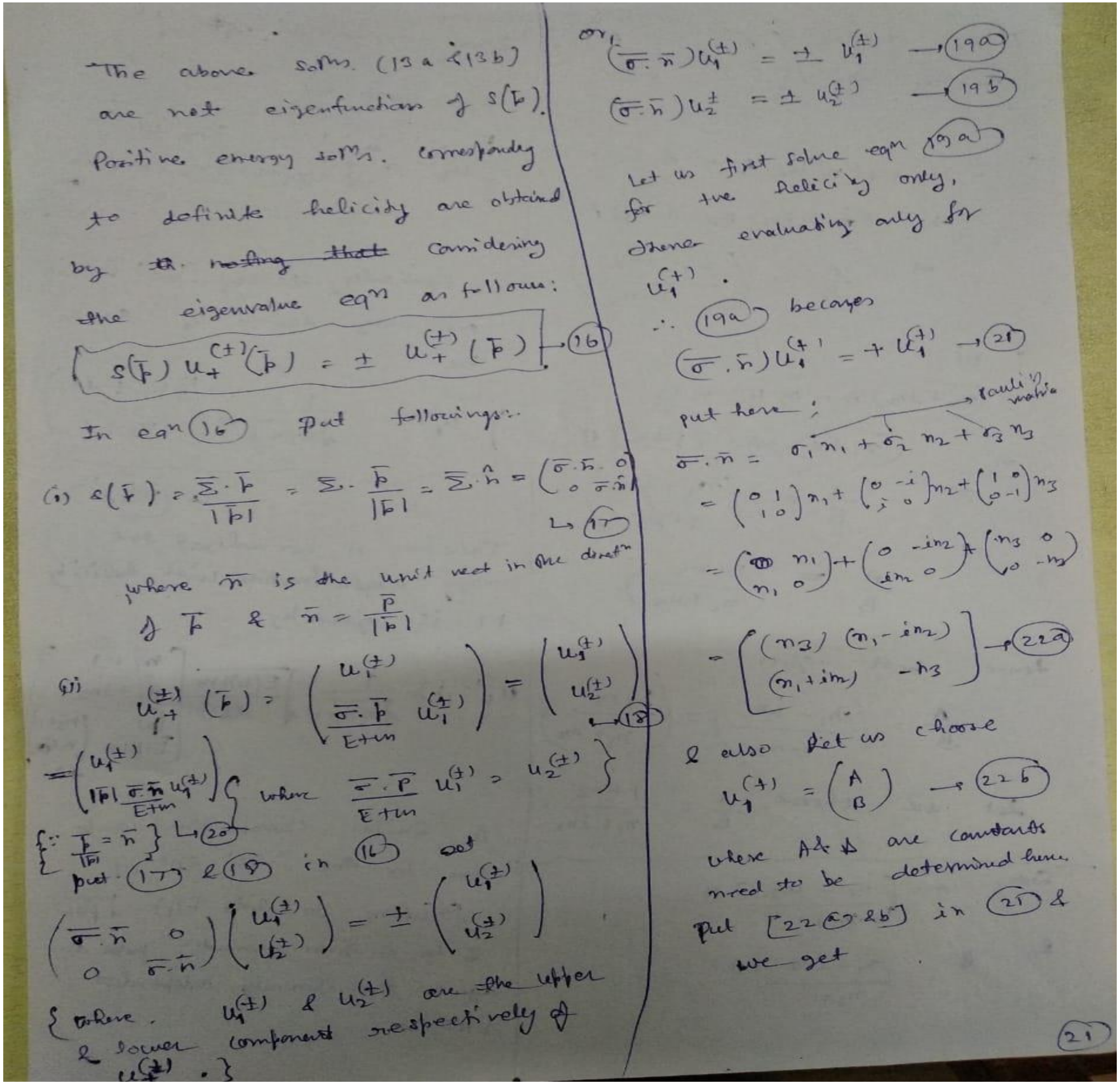
$$\Rightarrow \boxed{N(p) = \left(\frac{E+m}{2E} \right)^{1/2}} \rightarrow (14)$$

Put this in (13a) & (13b) & get the explicit form of linearly independent solⁿˢ for two +ve energy & momentum $\bar{p}$. (Put & find out)

It may be noted that these two solutions (13a) & (13b) are orthogonal to each other, i.e.,

$$\boxed{u_+^{(r)*}(p)\, u_+^{(s)}(p) = \delta_{rs} , \quad r, s = 1,2} \rightarrow (15)$$

Solution of Dirac Equation for free particles: Continue



The above solns. (13a & 13b) are not eigenfunctions of $S(\bar{p})$. Positive energy solns. corresponding to definite helicity are obtained by ~~th. noting that~~ considering the eigenvalue eqn as follows:

$$\boxed{S(\bar{p})\, u_+^{(\pm)}(\bar{p}) = \pm\, u_+^{(\pm)}(\bar{p})} \longrightarrow (16)$$

In eqn (16) put followings:-

(i) $S(\bar{p}) = \frac{\bar{\Sigma}\cdot\bar{p}}{|\bar{p}|} = \Sigma\cdot\frac{\bar{p}}{|\bar{p}|} = \bar{\Sigma}\cdot\hat{n} = \begin{pmatrix}\bar{\sigma}\cdot\bar{n} & 0\\ 0 & \bar{\sigma}\cdot\hat{n}\end{pmatrix} \longrightarrow (17)$

where $\bar{n}$ is the unit vect in the directn of $\bar{p}$ & $\bar{n} = \frac{\bar{P}}{|\bar{p}|}$

(ii) $u_{+}^{(\pm)}(\bar{p}) = \begin{pmatrix} u_1^{(\pm)} \\ \frac{\bar{\sigma}\cdot\bar{p}}{E+m}\, u_1^{(\pm)}\end{pmatrix} = \begin{pmatrix}u_1^{(\pm)}\\ u_2^{(\pm)}\end{pmatrix} \longrightarrow (18)$

$= \begin{pmatrix} u_1^{(\pm)} \\ |\bar{p}|\,\frac{\bar{\sigma}\cdot\bar{n}}{E+m}\, u_1^{(\pm)}\end{pmatrix} \longrightarrow (20)$ { where $\frac{\bar{\sigma}\cdot\bar{P}}{E+m}\, u_1^{(\pm)} = u_2^{(\pm)}$ }

{ ∵ $\frac{\bar{p}}{|\bar{p}|} = \bar{n}$ }

put (17) & (18) in (16) set

$$\begin{pmatrix}\bar{\sigma}\cdot\bar{n} & 0\\ 0 & \bar{\sigma}\cdot\bar{n}\end{pmatrix}\begin{pmatrix}u_1^{(\pm)}\\ u_2^{(\pm)}\end{pmatrix} = \pm\begin{pmatrix}u_1^{(\pm)}\\ u_2^{(\pm)}\end{pmatrix}$$

{ where. $u_1^{(\pm)}$ & $u_2^{(\pm)}$ are the upper & lower components respectively of $u^{(\pm)}$. }

or, $(\bar{\sigma}\cdot\bar{n})\, u_1^{(\pm)} = \pm\, u_1^{(\pm)} \longrightarrow (19a)$

$(\bar{\sigma}\cdot\bar{n})\, u_2^{\pm} = \pm\, u_2^{(\pm)} \longrightarrow (19b)$

Let us first solve eqn (19a) for the +ve helicity only, there evaluating only for $u_1^{(+)}$.

∴ (19a) becomes

$(\bar{\sigma}\cdot\bar{n})\, u_1^{(+)} = +\, u_1^{(+)} \longrightarrow (21)$

put here; Pauli's matrix

$\bar{\sigma}\cdot\bar{n} = \sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3$

$= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} n_1 + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} n_2 + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} n_3$

$= \begin{pmatrix}0 & n_1\\ n_1 & 0\end{pmatrix} + \begin{pmatrix}0 & -in_2\\ in_2 & 0\end{pmatrix} + \begin{pmatrix}n_3 & 0\\ 0 & -n_3\end{pmatrix}$

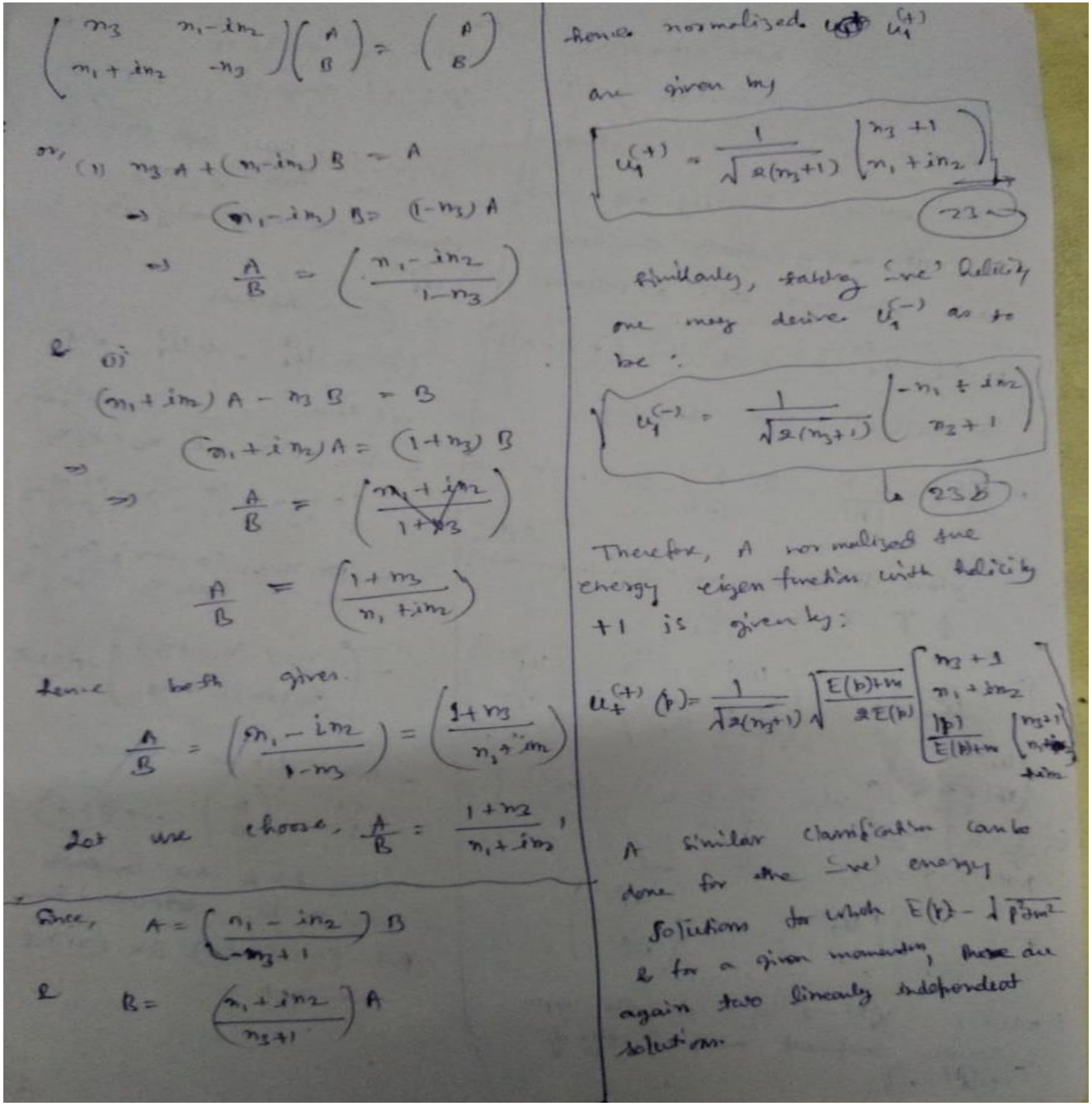
$= \begin{bmatrix}(n_3) & (n_1 - in_2)\\ (n_1 + in_2) & -n_3\end{bmatrix} \longrightarrow (22a)$

& also let us choose

$u_1^{(+)} = \begin{pmatrix}A\\ B\end{pmatrix} \longrightarrow (22b)$

where A & B are constants need to be determined here. Put [22(a) & b] in (21) & we get

Solution of Dirac Equation for free particles: Continue



$$\begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

or, (i) $n_3 A + (n_1 - in_2) B = A$

$$\Rightarrow (n_1 - in_2) B = (1 - n_3) A$$

$$\Rightarrow \frac{A}{B} = \left(\frac{n_1 - in_2}{1 - n_3} \right)$$

& (ii)

$$(n_1 + in_2) A - n_3 B = B$$

$$\Rightarrow (n_1 + in_2) A = (1 + n_3) B$$

$$\Rightarrow \frac{A}{B} = \left(\frac{n_1 + in_2}{1 + n_3} \right)$$

$$\frac{A}{B} = \left(\frac{1 + n_3}{n_1 + in_2} \right)$$

Hence both given

$$\frac{A}{B} = \left(\frac{n_1 - in_2}{1 - n_3} \right) = \left(\frac{1 + n_3}{n_1 + in_2} \right)$$

Let we choose, $\frac{A}{B} = \frac{1 + n_3}{n_1 + in_2}$,

Since, $A = \left(\frac{n_1 - in_2}{-n_3 + 1} \right) B$

& $B = \left(\frac{n_1 + in_2}{n_3 + 1} \right) A$

Hence normalized ~~u~~ $u_1^{(+)}$ are given by

$$\boxed{u_1^{(+)} = \frac{1}{\sqrt{2(n_3+1)}} \begin{pmatrix} n_3 + 1 \\ n_1 + in_2 \end{pmatrix}} \quad (23)$$

Similarly, taking '-ve' helicity one may derive $u_1^{(-)}$ as to be :

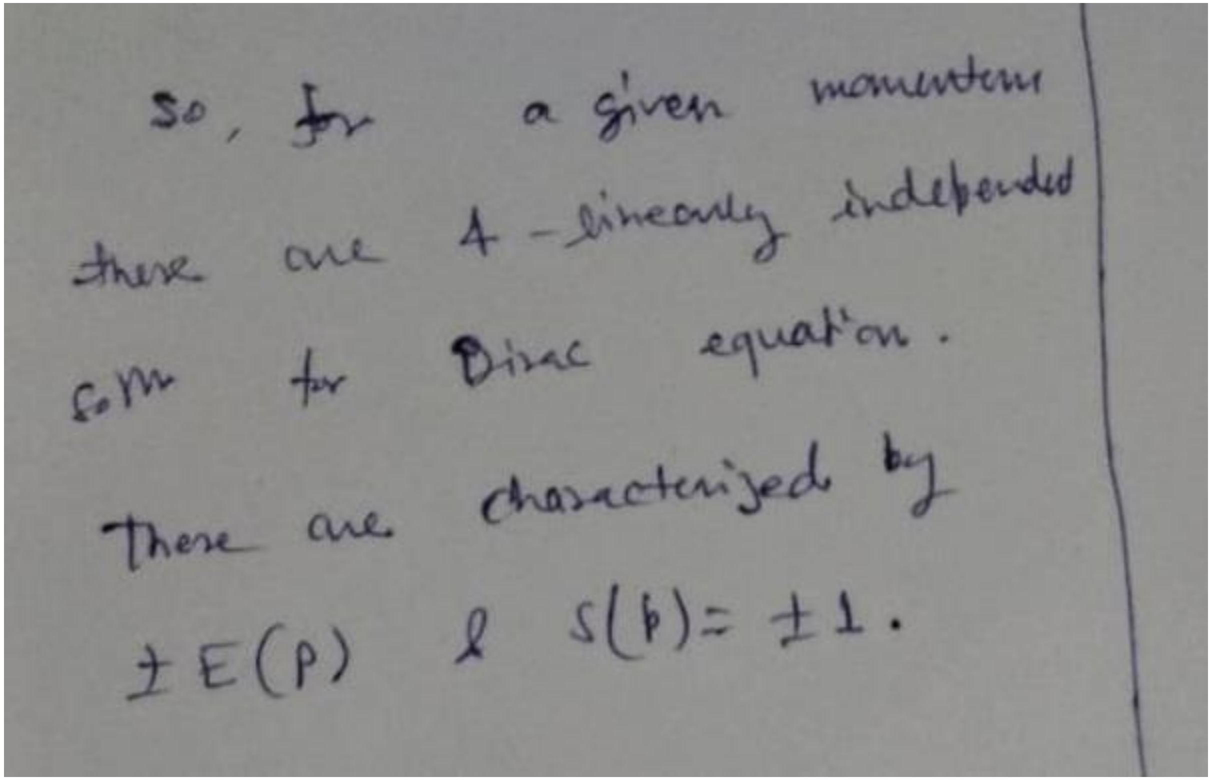
$$\boxed{u_1^{(-)} = \frac{1}{\sqrt{2(n_3+1)}} \begin{pmatrix} -n_1 + in_2 \\ n_3 + 1 \end{pmatrix}} \quad (23b)$$

Therefore, A normalized +ve energy eigen function with helicity +1 is given by:

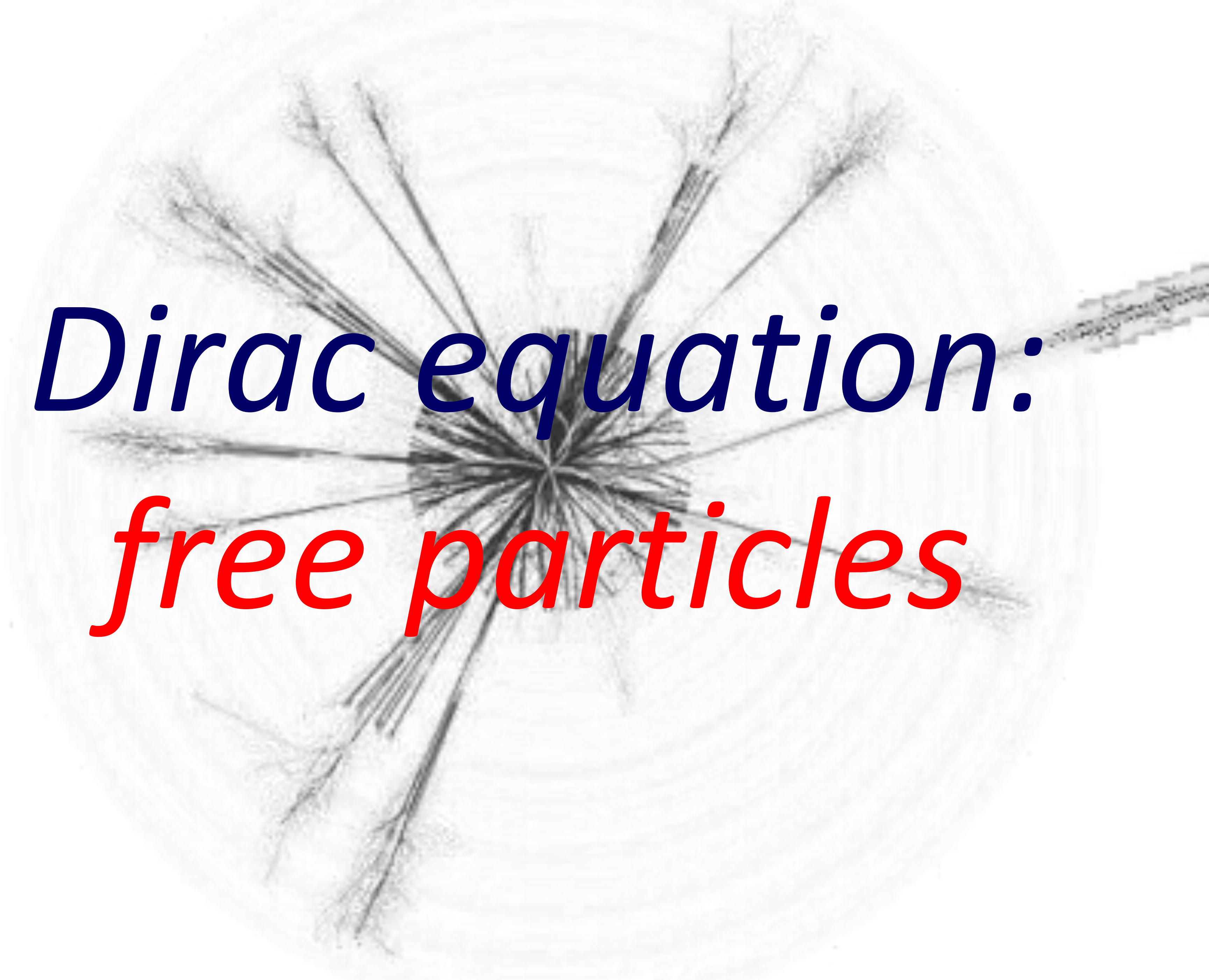
$$u_+^{(+)}(p) = \frac{1}{\sqrt{2(n_3+1)}} \sqrt{\frac{E(p)+m}{2E(p)}} \begin{bmatrix} n_3 + 1 \\ n_1 + in_2 \\ \frac{|p|}{E(p)+m} \begin{pmatrix} n_3 + 1 \\ n_1 + in_2 \end{pmatrix} \end{bmatrix}$$

A similar classification can be done for the '-ve' energy solutions for which $E(p) - \sqrt{p^2 + m^2}$ & for a given momentum, there are again two linearly independent solutions.

Solution of Dirac Equation for free particles: Continue



so, for a given momentum
there are 4 - linearly independed
form for Dirac equation.
These are characterized by
$\pm E(p)$ & $s(p) = \pm 1$.



Dirac equation:
free particles

Schrödinger – Klein-Gordon – Dirac

Quantum mechanical E & p operators:
$$\begin{cases} E = i\dfrac{\partial}{\partial t} \\ \vec{p} = -i\vec{\nabla} \end{cases}$$

$$\boxed{p^{\mu} = (E, \vec{p}) \rightarrow i\partial^{\mu} = i\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)}$$

You simply 'derive' the Schrödinger equation from classical mechanics:

$$E = \frac{p^2}{2m} \quad \rightarrow \quad i\frac{\partial}{\partial t}\phi = -\frac{1}{2m}\nabla^2\phi$$

Schrödinger equation

With the relativistic relation between E, p & m you get:

$$E^2 = \boldsymbol{p}^2 + m^2 \quad \rightarrow \quad \frac{\partial^2}{\partial t^2}\phi = \nabla^2\phi - m^2\phi$$

Klein-Gordon equation

The negative energy solutions led Dirac to try an equation with first order derivatives in time (like Schrödinger) <u>as well as in space</u>

$$i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi$$

Dirac equation

Does it make sense?

Also Dirac equation should reflect: $E^2=\vec{p}^2+m^2$

Basically squaring: $i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi = \vec{\alpha}\cdot\vec{p}\phi + \beta m\phi$

Tells you:

$$\begin{aligned}
\underbrace{(\vec{\alpha}\cdot\vec{p}+\beta mc)^2}_{E^2} &= (\alpha_i p_i+\beta mc)(\alpha_j p_j+\beta mc) \\
m^2 \longleftarrow \quad &= \beta^2 m^2 c^2 \longrightarrow \beta^2=1 \\
\vec{p}^2 \longleftarrow \quad & \quad + \sum_i \left[\alpha_i^2 p_i^2 + (\alpha_i\beta+\beta\alpha_i)p_i mc\right] \longrightarrow \alpha_i{}^2=1,\ \alpha_i\beta+\beta\alpha_i=0 \\
& \quad + \sum_{i>j}\left[(\alpha_i\alpha_j+\alpha_j\alpha_i)p_ip_j\right] \longrightarrow i\neq j:\ \alpha_i\,\alpha_j+\alpha_j\,\alpha_i=0
\end{aligned}$$

Properties of α_i and β

β and α can not be simple commuting numbers, but must be matrices

Because $\beta^2 = \alpha_i^2 = 1$, both β and α must have eigenvalues ±1

Since the eigenvalues are real (±1), both β and α must be Hermitean

$$\alpha_i^\dagger = \alpha_i \quad \text{en} \quad \beta^\dagger = \beta$$

$$A_{ij}B_{jk}C_{ki} = C_{ki}\,A_{ij}B_{jk} = B_{jk}C_{ki}\,A_{ij}$$

Both β and α must be traceless matrices: Tr(ABC) = Tr(CAB) = Tr(BCA)

$$\mathrm{Tr}(\alpha_i) \overset{\beta^2=1}{=} \mathrm{Tr}(\alpha_i\beta\beta) \overset{\text{cyclic}}{=} \mathrm{Tr}(\beta\alpha_i\beta) \overset{\substack{\text{anti}\\ \text{commutation}}}{=} -\mathrm{Tr}(\alpha_i\beta\beta) \overset{\beta^2=1}{=} -\mathrm{Tr}(\alpha_i) \text{ and hence } \mathrm{Tr}(\alpha_i)=0$$

You can easily show the dimension d of the matrices β, α to be even:

either: $$i \neq j: \; |\alpha_i\alpha_j| = |-\alpha_j\alpha_i| = (-1)^d|\alpha_j\alpha_i| = \begin{cases} -|\alpha_i\alpha_j|, & \text{d odd} \\ +|\alpha_i\alpha_j|, & \text{d even} \end{cases}$$

or: with eigenvalues ±1, matrices are only traceless in even dimensions

Explicit expressions for α_i and β

In 2 dimensions, you find at most 3 anti-commuting matrices,
Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In 4 dimensions, you can find 4 anti-commuting matrices,
numerous possibilities, Dirac-Pauli representation:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Any other set of 4 anti-commutating matrices will give same physics
(if the Dirac equation is to make any sense at all of course .....
and ... if it would not: we would not be discussing it here!)

Co-variant form: Dirac γ-matrices

$$i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi$$ ***does not look that Lorentz invariant***

Multiplying on the left with β and collecting all the derivatives gives:

$$m\phi = i\beta\frac{\partial}{\partial t}\phi + i\,\beta\vec{\alpha}\cdot\vec{\nabla}\phi \equiv i\gamma^{\mu}\partial_{\mu}\phi$$

note: $\partial_{\mu} = (\partial_t, +\vec{\nabla})$

Hereby, the Dirac γ-matrices are defined as:

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^k \equiv \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

And you can verify that: $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$

As well as: $$(\gamma^0)^2 = +1 \qquad (\gamma^k)^2 = -1$$

and: $$\gamma^{0\dagger} = +\gamma^0 \qquad \gamma^{k\dagger} = -\gamma^k$$

$\rightarrow \boldsymbol{\gamma^{\mu+} = \gamma^0\gamma^{\mu}\gamma^0}$

Co-variant form: Dirac γ-matrices

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$m\phi = i\gamma^{\mu}\partial_{\mu}\phi$ **with the Dirac γ-matrices defined as:**

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^k \equiv \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Warning!

This $m\phi = i\gamma^{\mu}\partial_{\mu}\phi$ ***notation is misleading,*** γ^{μ} ***is not a 4-vector!***

The γ^{μ} ***are just a set of four 4×4 matrices, which do no not transform at all i.e. in every frame they are the same, despite the*** μ***-index.***

The Dirac wave-functions (ϕ ***or*** ψ***), so-called ´spinors´ have interesting Lorentz transformation properties which we will discuss shortly.***

After that it will become clear why the notation with γ^{μ} ***is useful!***

& beautiful!

To make things even worse, we define:

$$\begin{cases} \gamma_0 = +\gamma^0 \\ \gamma_k = -\gamma^k \end{cases}$$

Spinors & (Dirac) matrices

$$\phi = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \qquad \phi^+ = \begin{pmatrix} * & * & * & * \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\gamma^\mu \phi = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \times \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$$

$$\gamma^\mu \phi^+ = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * & * \end{pmatrix} = \text{↯}$$

$$\phi^+ \gamma^\mu = \begin{pmatrix} * & * & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \end{pmatrix}$$

$$\phi\, \gamma^\mu = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \times \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} = \text{↯}$$

$$\phi^+ \phi = \begin{pmatrix} * & * & * & * \end{pmatrix} \times \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \end{pmatrix}$$

$$\phi\, \phi^+ = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \times \begin{pmatrix} * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

this one we will encounter later …

Dirac current & probability densities

Proceed analogously to Schrödinger & Klein-Gordon equations, but with Hermitean instead of complex conjugate wave-functions:

$$\begin{aligned} 0 &= -i\hbar\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - mc\psi^\dagger \\ &= -i\hbar\partial_0\psi^\dagger\gamma^0 + i\hbar\partial_k\psi^\dagger\gamma^k - mc\psi^\dagger \\ &\longrightarrow -i\hbar\partial_0\psi^\dagger\gamma^0\gamma^0 + i\hbar\partial_k\psi^\dagger\gamma^k\gamma^0 - mc\psi^\dagger\gamma^0 \\ &= -i\hbar\partial_0\psi^\dagger\gamma^0\gamma^0 - i\hbar\partial_k\psi^\dagger\gamma^0\gamma^k - mc\psi^\dagger\gamma^0 \\ &= -i\hbar\partial_\mu\psi^\dagger\gamma^0\gamma^\mu - mc\psi^\dagger\gamma^0 \\ (\bar\psi\equiv\psi^\dagger\gamma^0) &\longrightarrow -i\hbar\partial_\mu\bar\psi\gamma^\mu - mc\bar\psi \end{aligned}$$

$\longleftarrow \times\, \gamma^0$

Dirac equations for $\bar\psi$ **&** ψ **:**

$$\begin{cases} i\hbar(\partial_\mu\bar\psi)\gamma^\mu + mc\bar\psi = 0 & \longleftarrow \times\,\psi \\ \longrightarrow \times\,\bar\psi \quad i\hbar\gamma^\mu(\partial_\mu\psi) - mc\psi = 0 \end{cases}$$

Add these two equations to get:

Conserved 4-current: $0 = i\hbar(\partial_\mu\bar\psi)\gamma^\mu\psi + i\hbar\bar\psi\gamma^\mu(\partial_\mu\psi) = i\hbar\partial_\mu\left[\bar\psi\gamma^\mu\psi\right]$

$$j^\mu = \bar\psi\gamma^\mu\psi \begin{cases} j^0 = \bar\psi\gamma^0\psi = |\psi_0|^2 + |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \geq 0 \\ j^k = \bar\psi\gamma^k\psi \end{cases}$$

(exactly what Dirac aimed to achieve …)

Solutions: particles @ rest $\vec{p} = \vec{0}$

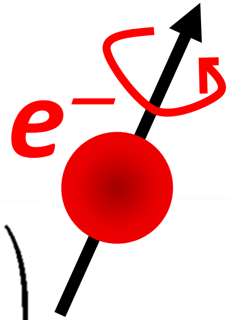
Dirac equation for $\vec{p} = \vec{0}$ is simple: $i\hbar\gamma^0\partial_0\psi - mc\psi = 0$

Solve by splitting 4-component in two 2-components: $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$

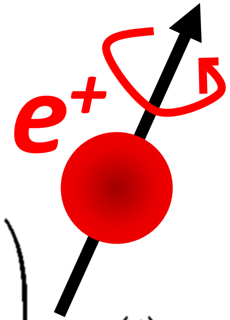
with $\partial_0 \equiv (1/c)\partial_t$ follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial\psi_A/\partial_t \\ \partial\psi_B/\partial_t \end{pmatrix} = -\frac{imc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

solutions:



$$\psi^{(1)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \psi^{(2)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$



$$\psi^{(3)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \psi^{(4)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Solutions: moving particles $\vec{p} \neq \vec{0}$

Dirac equation for $\vec{p} \neq \vec{0}$ less simple:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

Anticipate plane-waves:

$$\psi = u(p)e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{x})} = u(p)e^{-\frac{i}{\hbar}p\cdot x}$$

And again anticipate two 2-components:

$$u(p) = \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix}$$

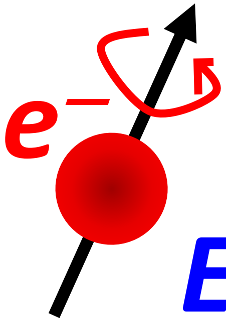
Plugging this in gives:

$$\begin{aligned} 0 &= (\gamma^\mu p_\mu - mc)\,u(p) = \left(\gamma^0 p_0 - \gamma^k p_k - mc\right)u(p) \\ &= \begin{pmatrix} E/c - mc & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E/c - mc \end{pmatrix} \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} \\ &= \begin{pmatrix} (E/c - mc)u_A(p) - \vec{p}\cdot\vec{\sigma}u_B(p) \\ \vec{p}\cdot\vec{\sigma}u_A(p) - (E/c + mc)u_B(p) \end{pmatrix} \end{aligned}$$

$$\Rightarrow \begin{cases} u_A(p) = \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})u_B(p) \\ u_B(p) = \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A(p) \end{cases}$$

Solutions: moving particles $\vec{p} \neq \vec{0}$

Solutions: pick $u_A(p)$ & calculate $u_B(p)$: $u_B(p) = \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A(p)$



E>0

$u_A = \begin{pmatrix}1\\0\end{pmatrix} \rightarrow$

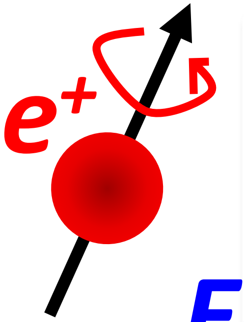
$$\psi^{(1)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix}1\\0\\\frac{cp_z}{E+mc^2}\\\frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}$$

$u_A = \begin{pmatrix}0\\1\end{pmatrix} \rightarrow$

$$\psi^{(2)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix}0\\1\\\frac{c(p_x-ip_y)}{E+mc^2}\\\frac{-cp_z}{E+mc^2}\end{pmatrix}$$

In limit $\vec{p} \rightarrow \vec{0}$ you retrieve the E>0 solutions, hence these are $\vec{p} \neq \vec{0}$ electron solutions

Similarly: pick $u_B(p)$ & calculate $u_A(p)$: $u_A(p) = \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})u_B(p)$



E<0

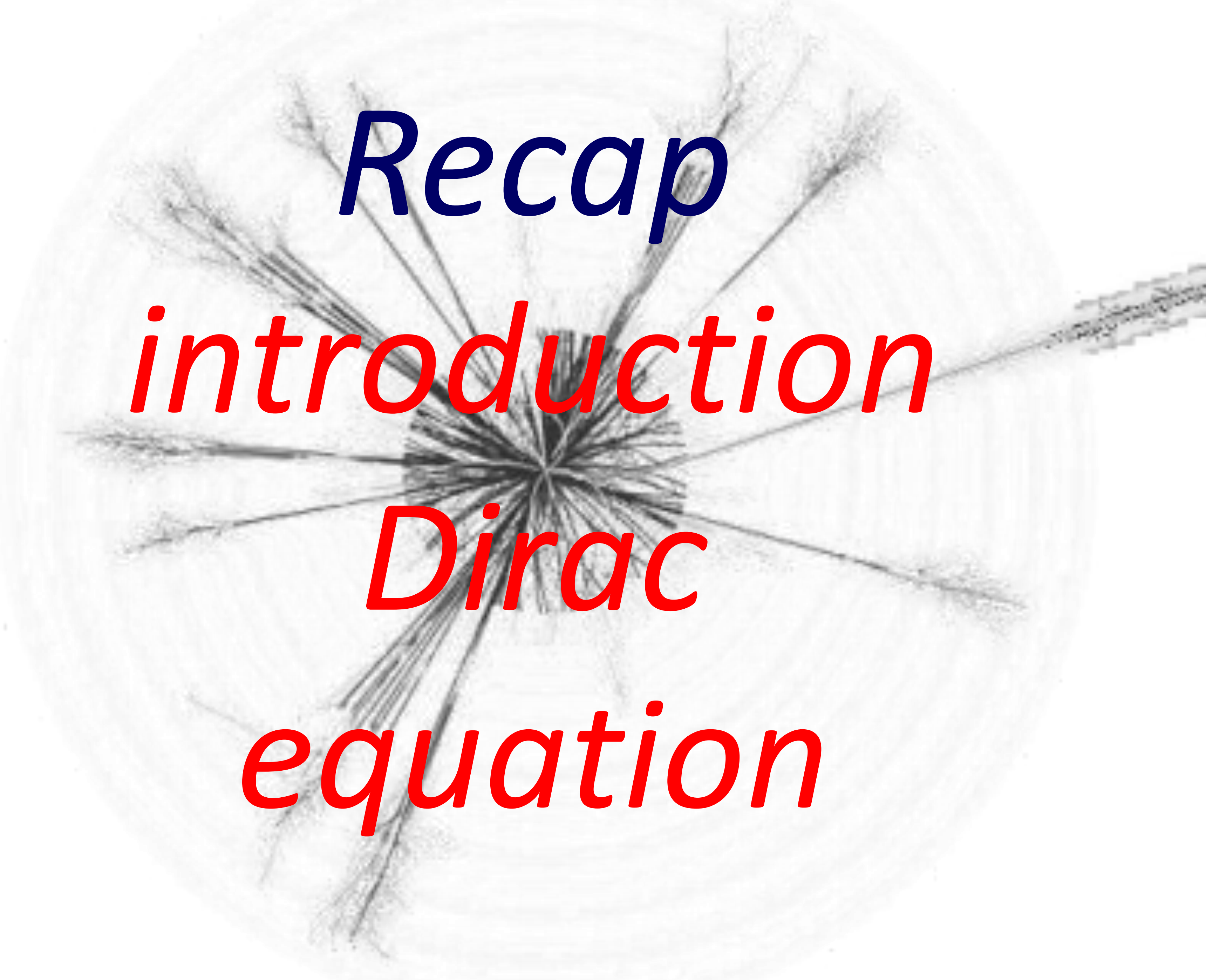
$u_B = \begin{pmatrix}1\\0\end{pmatrix} \rightarrow$

$$\psi^{(3)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix}\frac{cp_z}{E-mc^2}\\\frac{c(p_x+ip_y)}{E-mc^2}\\1\\0\end{pmatrix}$$

$u_B = \begin{pmatrix}0\\1\end{pmatrix} \rightarrow$

$$\psi^{(4)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix}\frac{c(p_x-ip_y)}{E-mc^2}\\\frac{-cp_z}{E-mc^2}\\0\\1\end{pmatrix}$$

In limit $\vec{p} \rightarrow \vec{0}$ you retrieve the E<0 solutions, hence these are $\vec{p} \neq \vec{0}$ positron solutions



Recap

introduction Dirac equation

Dirac equation

$\partial^{\mu} = (\partial_t, -\vec{\nabla})$

From: $E^2 = \vec{p}^2 + m^2$ & classical → QM 'transcription': $\begin{cases} E = i\dfrac{\partial}{\partial t} \\ \vec{p} = -i\vec{\nabla} \end{cases}$

We found: $$i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi = \vec{\alpha}\cdot\vec{p}\phi + \beta m\phi$$

With β, α_1, α_2 & α_3 (4×4) matrices, satisfying:

$$\boxed{E^2 \overset{?}{=} \vec{p}^2 + m^2}$$

$$\underbrace{(\vec{\alpha}\cdot\vec{p} + \beta mc)^2}_{E^2} = (\alpha_i p_i + \beta mc)(\alpha_j p_j + \beta mc)$$

$$m^2 \longleftarrow \quad = \beta^2 m^2 c^2 \longrightarrow \beta^2 = 1$$

$$\vec{p}^2 \longleftarrow \quad + \sum_i \left[\alpha_i^2 p_i^2 + (\alpha_i\beta + \beta\alpha_i)p_i mc\right] \longrightarrow \alpha_i^2 = 1, \quad \alpha\beta + \beta\alpha = 0$$

$$+ \sum_{i>j} \left[(\alpha_i\alpha_j + \alpha_j\alpha_i)p_i p_j\right] \longrightarrow i \neq j\!: \ \alpha_i\alpha_j + \alpha_j\alpha_i = 0$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Co-variant form: Dirac γ-matrices

Dirac's original form does not look covariant: $i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi$

Multiplying on the left with β and collecting all the derivatives gives covariant form:

$$m\phi = i\beta\frac{\partial}{\partial t}\phi + i\,\beta\vec{\alpha}\cdot\vec{\nabla}\phi \equiv i\gamma^{\mu}\partial_{\mu}\phi$$

note: $\boldsymbol{\partial_\mu = (\partial_t, +\vec{\nabla})}$

With Dirac γ-matrices defined as: $\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\gamma^k = \beta\alpha^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

From the properties of β, α_1, α_2 & α_3 follows: $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$

$$(\gamma^0)^2 = +1$$
$$(\gamma^k)^2 = -1$$

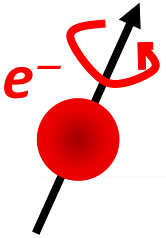
$$\gamma^{0\dagger} = +\gamma^0$$
$$\gamma^{k\dagger} = -\gamma^k$$

$$\rightarrow \boldsymbol{\gamma^{\mu+} = \gamma^0\gamma^{\mu}\gamma^0}$$

Dirac particle solutions: spinors

Ansatz solution: $\psi = \begin{bmatrix} u_A(p) \\ u_B(p) \end{bmatrix} e^{-ip\cdot x} \longrightarrow$ **Dirac eqn.:** $\begin{cases} u_A(p) = \dfrac{\vec{p}\cdot\vec{\sigma}}{E-m} u_B(p) \\ u_B(p) = \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A(p) \end{cases}$

$\vec{p} = \vec{0}$ solutions:



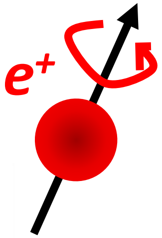
$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\psi^{(1)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\psi^{(2)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

spin ½ electrons **E>0**



$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\psi^{(3)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\psi^{(4)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

spin ½ positrons **E<0**

$\vec{p} \neq \vec{0}$ solutions:

$$\psi^{(1)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

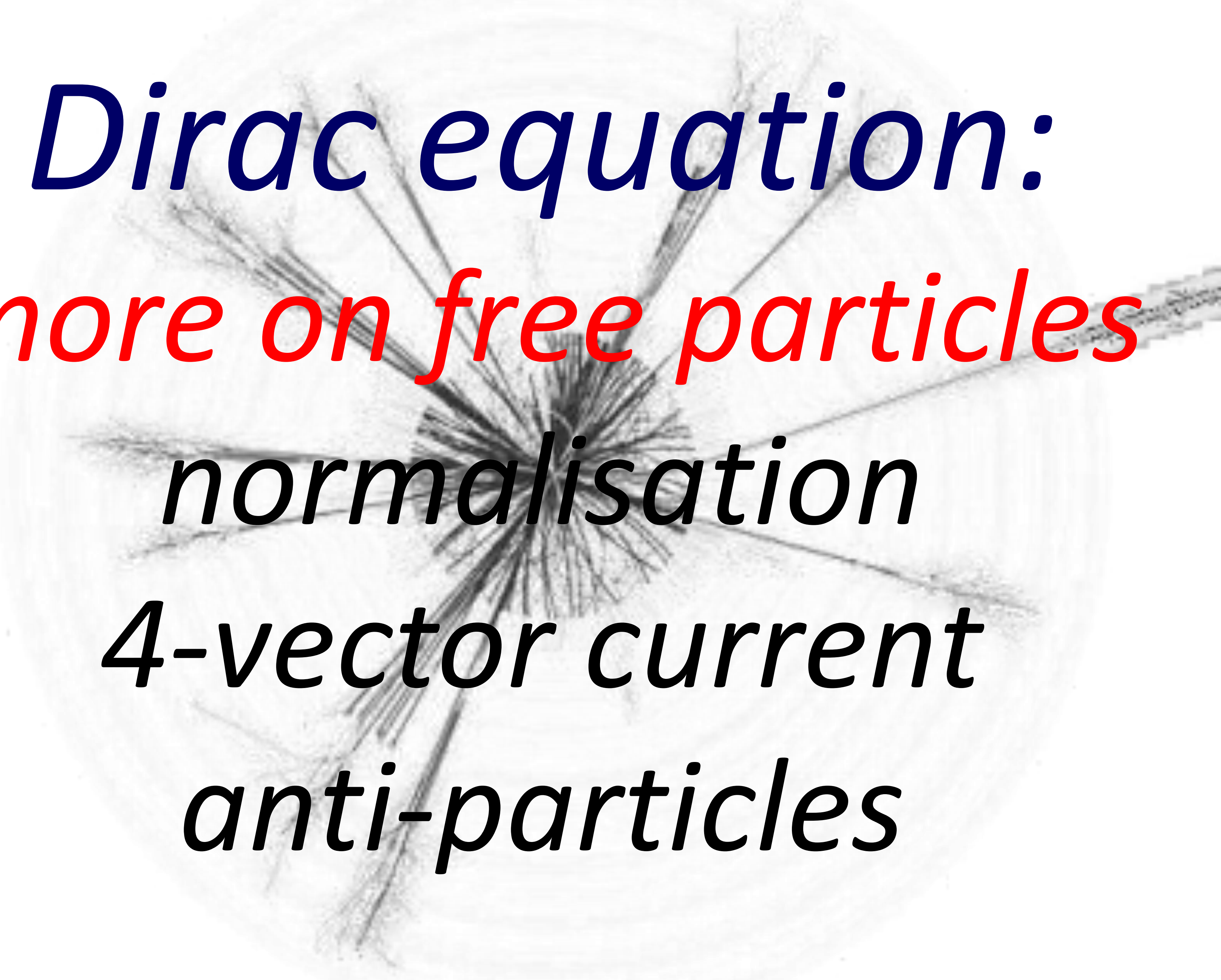
$$\psi^{(2)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}$$

$$\psi^{(3)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$\psi^{(4)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} \frac{c(p_x-ip_y)}{E-mc^2} \\ \frac{-cp_z}{E-mc^2} \\ 0 \\ 1 \end{pmatrix}$$

Dirac equation:

more on free particles



normalisation

4-vector current

anti-particles

sorry for the c's

One more look at $\vec{p}\cdot\vec{\sigma}$

The conditions:
$$\left\{\begin{array}{lcl} u_A(p) & = & \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})u_B(p) \\ u_B(p) & = & \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A(p) \end{array}\right.$$

Imply:
$$u_A(p) = \frac{c^2}{E^2-m^2c^4}(\vec{p}\cdot\vec{\sigma})^2 u_A(p)$$

$$\Rightarrow 1 = \frac{c^2}{E^2-m^2c^4}(\vec{p}\cdot\vec{\sigma})^2$$

$$\Rightarrow \ p^2c^2 = E^2 - m^2c^4$$

i.e. energy-momentum relation, as expected

Check this:
$$(\vec{p}\cdot\vec{\sigma}) = p_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} p_z & (p_x - ip_y) \\ (p_x + ip_y) & -p_z \end{pmatrix} \Rightarrow (\vec{p}\cdot\vec{\sigma})^2 = \begin{pmatrix} p_z^2 + (p_x - ip_y)(p_x + ip_y) & \dots \\ \dots & \dots \end{pmatrix} = \vec{p}^2$$

Normalisation of the Dirac spinors

Just calculate it!:

Spinors 1 & 2, E>0:

$$\psi^{(1)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

$$\begin{aligned} \psi^\dagger\psi &= 1+\frac{p_x^2c^2+p_y^2c^2+p_z^2c^2}{(E+mc^2)^2} \\ &= 1+\frac{E^2-m^2c^4}{(E+mc^2)^2} \\ &= 1+\frac{E-mc^2}{E+mc^2}=\frac{2E}{E+mc^2}=\frac{2|E|}{|E|+mc^2} \rightarrow N=\sqrt{|E|+mc^2} \end{aligned}$$

To normalize @ 2E particles/unit volume

Spinors 3 & 4, E<0:

$$\psi^{(3)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \psi^\dagger\psi &= 1+\frac{p_x^2c^2+p_y^2c^2+p_z^2c^2}{(E-mc^2)^2} \\ &= 1+\frac{E^2-m^2c^4}{(E-mc^2)^2} \\ &= 1+\frac{E+mc^2}{E-mc^2}=\frac{2E}{E-mc^2}=\frac{2|E|}{|E|+mc^2} \rightarrow N=\sqrt{|E|+mc^2} \end{aligned}$$

To normalize @ 2E particles/unit volume

Thanks