Finite Element Method: Introduction

Lecture01

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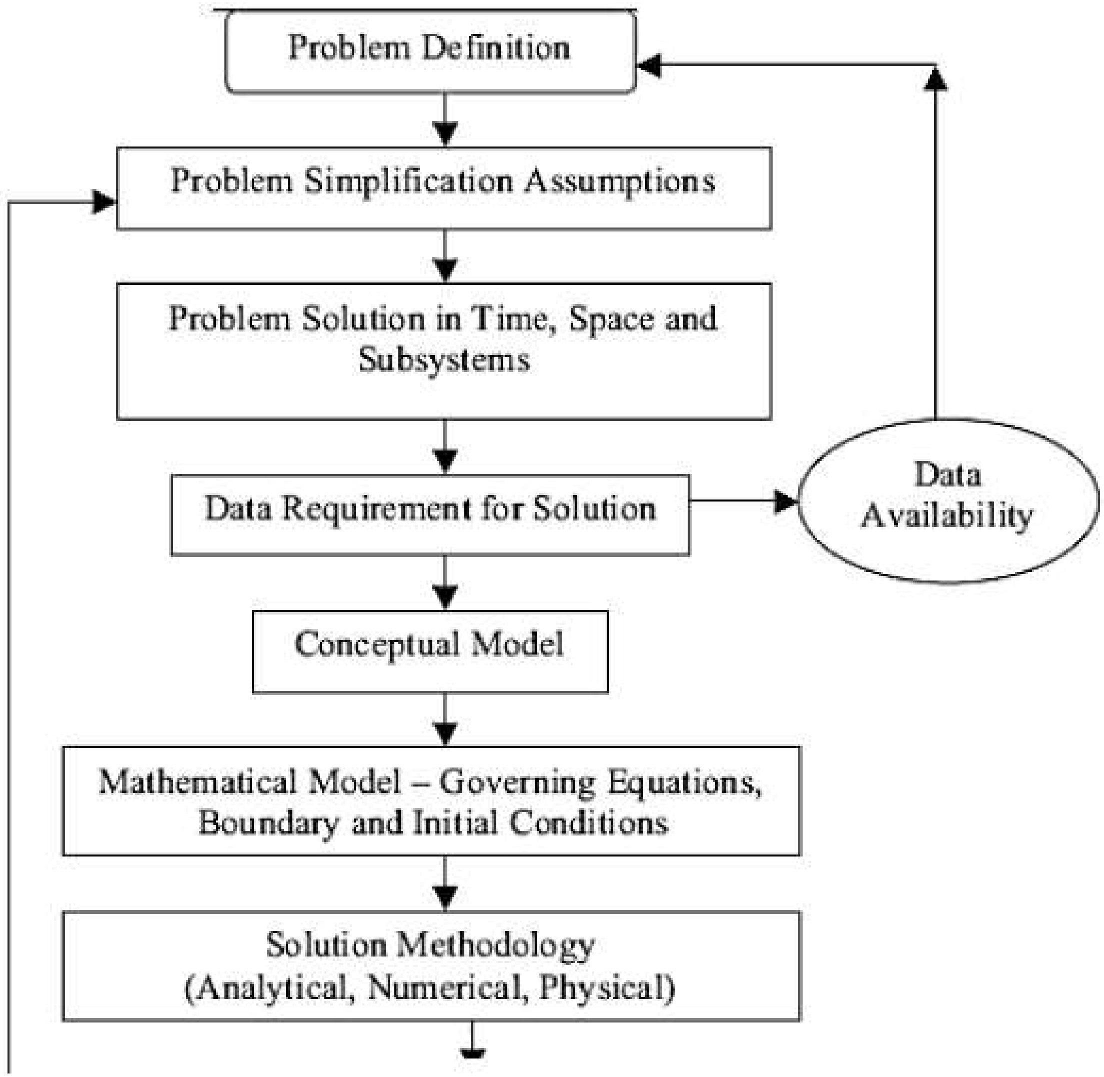
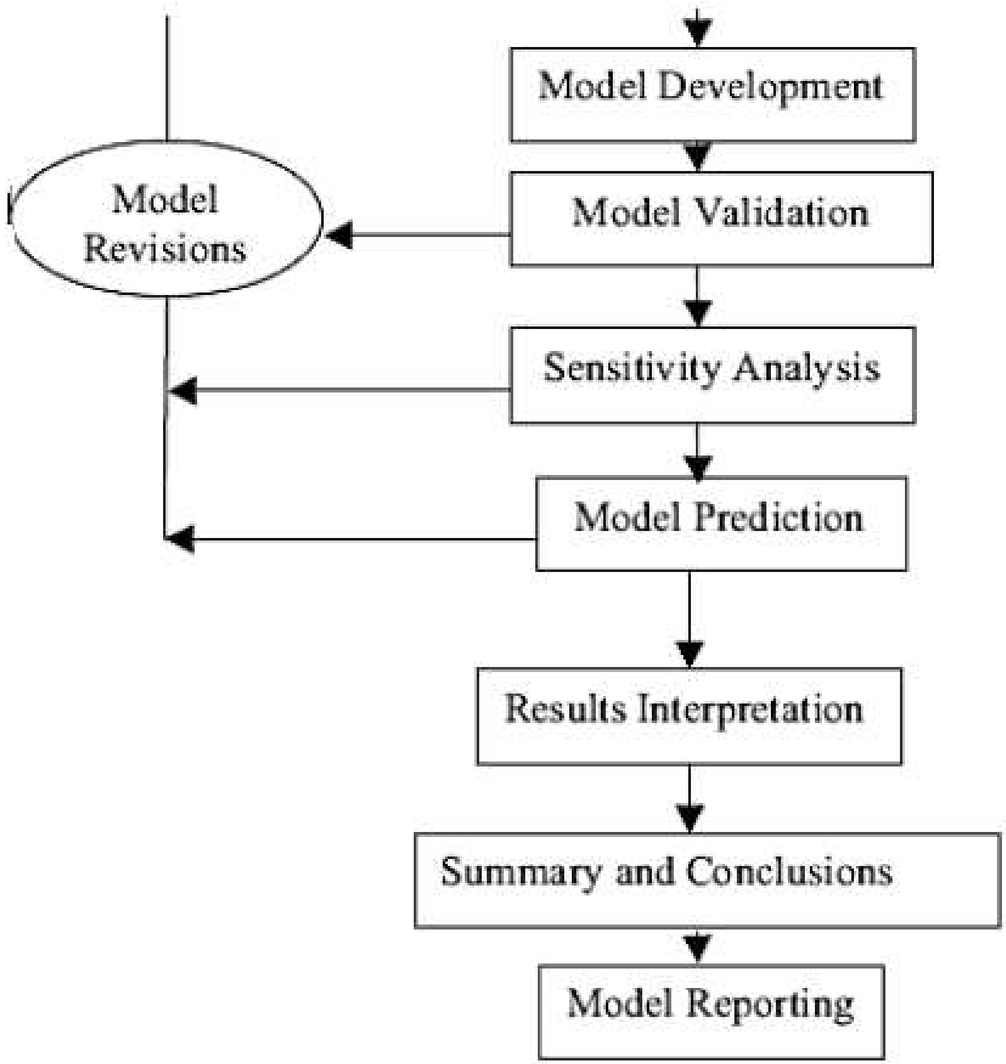
Finite Element Method

Introduction

For studying physical phenomena or engineering problems, engineers and scientists are involved with two major tasks:

- **Mathematical formulation of the physical problem:** The behaviour of the problem is expressed or modeled by means of integro -differential equations. Such equations are quite complicated and are known as behaviour /governing equations.
- **Numerical analysis of the mathematical model:** It is very difficult or impossible to solve the complicated integro -differential governing equations by conventional methods of mathematics. Hence numerical methods which yield approximate solutions are adopted


Problem Definition
Problem Simplification Assumptions
Problem Solution in Time, Space and Subsystems
Data Requirement for Solution
Data Availability
Conceptual Model
Mathematical Model – Governing Equations, Boundary and Initial Conditions
Solution Methodology (Analytical, Numerical, Physical)


Model Development
Model Validation
Model Revisions
Sensitivity Analysis
Model Prediction
Results Interpretation
Summary and Conclusions
Model Reporting

Contd.

Introduction

- Many numerical methods of solving the behavoiur /governing equations have been tried. But most of them have their own restrictions. The Finite Element Method (FEM) is the only numerical method which has got less restrictions of usage. In fact it can be successfully applied to almost all fields of engineering like structural engineering, thermal problems, fluid flow, electrical field, magnetism, acoustics, earth quake analysis, seepage problems, soil mechanics, etc
- The basis of Finite Element Method (FEM) is ‘ Discretization ’ i.e., to represent the region (continuum) of the problem by an assemblage of finite number of standard shaped sub divisions known as ‘Elements’. Finite number of elements is a requirement of the numerical method. These elements are inter connected to each other at common points known as ‘Nodes’. The properties (geometrical and material) of the elements are first established in the form of ‘Element Equations’.
- They are then assembled to obtain a very large set of simultaneous equations known as ‘Global Equations’, which closely represents the behaviour of the whole problem. Boundary conditions and loads are applied to these global equations. Using a suitable matrix method the global equations are solved to obtain unknown nodal displacements. The element stresses and strains are then obtained from the evaluated nodal displacements. Hence, FEM with the help of numerical procedures produces a solution, which is close to the exact solution, Hence, FEM provides an approximate solution
- Boundary Value Problems: Most of the problems in engineering are Boundary Value Problems. These are problems where the values of the unknown variables are known at some region of the boundary of the problems. Example (1): In a cantilever beam problem, we intend to find the unknown variables (deflection and slope) at all points along the axis of the beam. But we know the values of deflection and slope at the fixed end which is zero each. Example (2): In a heat transfer problem of a furnace wall we intend to find the temperature distribution in the wall, i.e., temperature is the unknown variable. But we know the temperatures at the inner surface and the outside ambient surface of furnace walls.

Why FEM is used widely to solve problems of engineering:?

Why FEM is used widely to solve problems of engineering In all the problems of engineering, we may require to find the value of the dependent variable at any specified point in the continuum. For this, the governing differential equations must be solved to get the value of the dependent variable. But, in actual practice, engineering problems involve complicated geometries (continuums), loadings and varying material properties. Due to this, it may be impossible to specify the boundary conditions, consider material properties and solve the governing differential equation. In such a situation, we go for numerical methods such as "Finite Element Method" (FEM) to get approximate but acceptable solutions

Definition of FEM:

The Finite Element Method is a numerical method for solving problems of engineering and mathematical physics where their behaviour /governing equations are expressed by integral or differential equations. The FEM formulation of the problem results in a set of simultaneous algebraic equations for solution, instead of requiring the solution of the governing differential equation. This yields approximate values of the variables at discrete points in the continuum

Finite Element Method – What is it?:

The Finite Element Method (FEM) is a numerical method of solving systems of partial differential equations (PDEs) It reduces a PDE system to a system of algebraic equations that can be solved using traditional linear algebra techniques. In simple terms, FEM is a method for dividing up a very complicated problem into small elements that can be solved in relation to each other.

Course Contents

Unit-I

Introduction: Historical background,basic concepts of FEM, Comparison with Finite Difference Method, Advantages and limitations, Different approaches in Finite Element Method-Direct, Variational approach, Weighted Residual methods.

Unit-II

Direct Problems- Spring, Hydraulic Network; Resistance Network and Truss Systems

Finite element analysis of 1-D problems: formulation by different approaches (direct, potential energy and Galerkin); Derivation of elemental equations and their assembly, solution and its postprocessing. Applications in heat transfer, fluid mechanics and solid mechanics. Bending of beams, analysis of truss and frame.

Unit-III

Finite element analysis of 2-D problems:Finite Element modelling of single variable problems, triangular and rectangular elements; Applications in heat transfer, fluid mechanics and solid mechanics.

Unit-IV

Numerical considerations: numerical integration, error analysis, mesh refinement. Plane stress and plane strain problems; Bending of plates; Eigen value and time dependent problems; Discussion about preprocessors, postprocessors and finite element packages.

Course objectives

- To develop the ability to generate the governing finite element equations for systems governed by partial differential equations.
- To understand the use of the basic finite elements for structural applications using truss, beam, frame and plane elements;
- To understand the application and use of the finite element method for heat transfer problems.
- To demonstrate the ability to evaluate and interpret finite element method analysis results for design and evaluation purposes.
- To develop a basic understanding of the limitations of the finite element method and understand the possible error sources in its use.

Recommended Books

- Finite Element Method – Y. M. Desai, T. I. Eldho and A. H. Shah (Pearson)
- An Introduction to Finite Element Method – J. N. Reddy (Tata McGraw Hill).
- Finite Element Procedure in Engineering Analysis - K.J. Bathe (Tata McGraw Hill). (New Central book Agency)
- Concepts and Application of Finite Element Analysis- R.D. Cook, D.S. Malcus and M.E. Plesha (John Wiley)
- Introduction to Finite Elements in Engineering- T.R Chandrupatla and A.D. Belegundu (Prentice Hall India)
- The Finite Element Method – O.C. Zienkiewicz and R.L. Taylor (Tata McGraw Hill).
- Numerical Methods– E. Balagurswamy (Tata Mc Graw Hill)

Brief History

- **Lord John William Strutt Rayleigh (late 1800s), developed a method for predicting the first natural frequency of simple structures. It assumed a deformed shape for a structure and then quantified this shape by minimizing the distributed energy in the structure.**
- **Ritz ended this into a method, now known as the Rayleigh-Ritz method, for predicting the stress and displacement behavior of structures.**
- **In 1943, Richard Courant proposed breaking a continuous system into triangular segments.**
- **In the 1950s, a team form Boeing demonstrated that complex surfaces could be analyzed with a matrix of triangular shapes.**
- **Dr. Ray Clough coined the term “finite element” in 1960. The 1960s saw the true beginning of commercial FEA as digital computers replaced analog ones with the capability of thousands of operations per second.**

Brief History

- In the early 1960s, the MacNeal-Schwendle Corporation (MSC) develop a general purpose FEA code. This original code had a limit of 68,000 degrees of freedom. When the NASA contract was complete, MSC continued development of its own version called MSC/NASTRAN, while the original NASTRAN become available to the public and formed the basis of dozens of the FEA packages available today. Around the time MSC/NASTRAN was released, ANSYS, MARC, and SAP were introduced.
- By the 1970s, Computer aided design (CAD) was introduced later in the decade.
- Standards such as IGES and DXF. Permitted limited geometry transfer between the systems.

Brief History

- In the 1980s,CAD progressed from a 2D drafting tool to a 3D surfacing tool, and then to a 3D in the 1980s, the use of FEA and CAD on the same workstation with developing geometry modeling system. Design engineers began to seriously consider incorporating FEA into the general product design process.
- As the 1990s draw to a place, the PC platform has become a major force in high end analysis. The technology has become to accessible that it is actually being "hidden" inside CAD packages.

FEM Packages

- NASTRAN
- ANSYS
- ABAQUS
- LS-DYNA
- DEFORM

References

- Finite Element Method – Y. M. Desai, T. I. Eldho and A. H. Shah (Pearson)
- An Introduction to Finite element method- J. N. Reddy (TMH)


THANK YOU

Finite Element Method: Basic Steps

Lecture02

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APROXIMATING METHODS

- It is possible to obtain exact analytical solutions for few simplified engineering problems. The exact solution can be obtained by direct integration of the concerned differential equations. Generally, the direct integration is made possible by: (i) separation of variables; (ii) similarity solutions; or (iii) Fourier and Laplace transformations. Since most engineering problems are complex in nature, number of problems having exact solutions is severely limited. Hence, approximate solutions are generally sought.
- Generally used approximating methods are

1. Perturbation Techniques 2. Power Series Solutions

3. Probability Methods 4. Finite Difference Method (FDM)

5. Method of Weighted Residuals (MWR) 6. Rayleigh–Ritz Method 7. Finite Element Method (FEM)

Brief History of FEM

- Basic idea of FEM originated from analysis procedure used in framed structures like trusses, aircraft structural analysis and flow network analysis. For the origin and development of FEM, there are different perspectives from the viewpoint of a mathematician, a physicist and an engineer. From the mathematician's perspective, solutions to boundary value problems of continuum mechanics were sought by finding approximate upper and lower bounds for eigenvalues. The physicists were trying to solve continuum mechanics problems by means of piecewise approximating functions. The engineers were investigating methodologies for the solution of complex aero-elasticity problems such as stiffness of shell type structures reinforced with ribs.

In the 1930s, when civil engineers dealt with truss analysis, they identified the solution procedure by solving the component stresses and deflections as well as the overall strength of the system. They recognized the truss as an assembly of members of rods whose force deflection characteristics can be easily obtained. By combining these individual characteristics using laws of equilibrium and solving the resulting system of equations, the unknown forces and deflections for the overall truss were obtained. Efforts of mathematicians, physicists and engineers finally resulted in the development of basic ideas of finite element method in 1940s.

Hrenikoff (1941) proposed the 'frame work method' for the solution of elasticity problems. On the other hand, Courant (1943) presented an assemblage of piecewise polynomial interpolation over triangular elements and the principle of minimum potential energy to solve torsion problems. Some mathematical aspects related to eigenvalues were developed for boundary value problems by Poyla (1954), Hersch (1955) and Weinberger (1958). Foundation of finite element was laid by Argyris in 1955 through his book on '*Energy Theorems and Structural Analysis*'. Turner et al. (1956) developed the stiffness matrices for truss, beam and other elements for engineering analysis of structures.

For the first time in 1960, the terminology 'Finite Element Method' was used by Clough (1960) in his paper on plane elasticity. In 1960s, a large number of papers appeared related to the applications and developments of the finite element method. Engineers applied FEM for stress analysis, fluid flow problems and heat transfer. A number of international conferences related to FEM were organized and the method got established. The first book on FEM was published by Zienkiewicz and Cheung in 1967. With the advent of digital computers and finding the suitability of FEM in fast computing for many engineering problems, the method became very popular among scientists, engineers and mathematicians. By now, a large number of research papers, proceedings of international conferences and short-term courses and books have been published on the subject of FEM. Many software packages are also available to deal with various types of engineering problems. As a result, FEM is the most acceptable and well-established numerical method in engineering sciences.

- In the 1980s,CAD progressed from a 2D drafting tool to a 3D surfacing tool, and then to a 3D in the 1980s, the use of FEA and CAD on the same workstation with developing geometry modeling system. Design engineers began to seriously consider incorporating FEA into the general product design process.
- As the 1990s draw to a place, the PC platform has become a major force in high end analysis. The technology has become to accessible that it is actually being “hidden” inside CAD packages.
- By now, a large number of research papers, proceedings of international conferences and short-term courses and books have been published on the subject of FEM. By the mid of 1990s roughly 40,000 papers and books about FE and its applications had been published. Many software packages are also available to deal with various types of engineering problems. As a result, FEM is the most acceptable and well-established numerical method in engineering sciences.

  - FEM/FEA is the most widely applied computer simulation method in engineering
  - Closely integrated with CAD/CAM applications

FEM Packages

- NASTRAN
- ANSYS
- ABAQUS
- LS-DYNA
- DEFORM
- I-DEAS
- COSMOS
- ALGOR
- PATRAN
- HYPER MESH

APROXIMATING METHODS

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- Generally used approximating methods are

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Finite Element Method

The finite element method is a numerical approach by which governing differential equations can be solved in an approximate manner.

$$\text{Physical Phenomena} \xrightarrow{\text{Model}} \text{Differential Equation} \xrightarrow{\text{Approximation}} \text{Finite Element Equations}$$

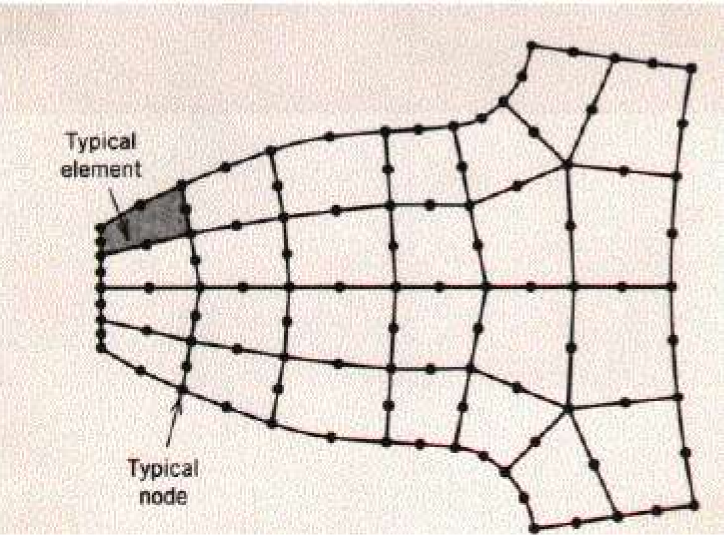
Basic Steps

- First, the governing differential equation of the problem is converted into an integral form. These are two techniques to achieve this : (i) Variational Technique and (ii) Weighted Residual Technique. In variational technique, the calculus of variation is used to obtain the integral form corresponding to the given differential equation. This integral needs to be minimized to obtain the solution of the problem. For structural mechanics problems, the integral form turns out to be the expression for the total potential energy of the structure. In weighted residual technique, the integral form is constructed as a weighted integral of the governing differential equation where the weight functions are known and arbitrary except that they satisfy certain boundary conditions. To reduce the continuity requirement of the solution, this integral form is often modified using the divergence theorem. This integral form is set to zero to obtain the solution of the problem. For structural mechanics problems, if the weight function is considered as the virtual displacement, then the integral form becomes the expression of the virtual work of the structure.

Basic Steps

Contd.

In the second step, the domain of the problem is divided into a number of parts, called as elements. For one dimensional (1-D) problems, the elements are nothing but line segments having only length and no shape. For problems of higher dimensions, the elements have both the shape and size. For two- dimensional (2D) or axisymmetric problems, the elements used are triangles, rectangles and quadrilateral having straight or curved boundaries. Curved sided elements are good choice when the domain boundary is curved. For three-dimensional (3-D) problems, the shapes used are tetrahedron and parallelepiped having straight or curved surfaces. Division of the domain into elements is called a mesh.



Basic Steps

Contd.

In this step, over a typical element, a suitable approximation is chosen for the primary variable of the problem using interpolation functions (also called as shape functions) and the unknown values of the primary variable at some preselected points of the element, called as the nodes. Usually polynomials are chosen as the shape functions. For 1-D elements, there are at least 2 nodes placed at the end-points. Additional nodes are placed in the interior of the element. For 2-D and 3-D elements, the nodes are placed at the vertices (minimum 3 nodes for triangles, minimum 4 nodes for rectangles, quadrilaterals and tetrahedral and minimum 8 nodes for parallelepiped shaped elements). Additional nodes are placed either on the boundaries or in the interior. The values of the primary variable at the nodes are called as the degrees of freedom.

Basic Steps

Contd.

To get the exact solution, the expression for the primary variable must contain a complete set of polynomials (i.e., infinite terms) or if it contains only the finite number of terms, then the number of elements must be infinite. In either case, it results into an infinite set of algebraic equations. To make the problem tractable, only a finite number of elements and an expression with only finite number of terms are used. Then, we get only an approximate solution. (Therefore, the expression for the primary variable chosen to obtain an approximate solution is called an approximation). The accuracy of the approximate solution, however, can be improved either by increasing the number of terms in the approximation or the number of elements.

Basic Steps

Contd.

- **In the fourth step, the approximation for the primary variable is substituted into the integral form. If the integral form is of variational type, it is minimized to get the algebraic equations for the unknown nodal values of the primary variable.**
- **If the integral form is of the weighted residual type, it is set to zero to obtain the algebraic equations. In each case, the algebraic equations are obtained element wise first (called as the element equations) and then they are assembled over all the elements to obtain the algebraic equations for the whole domain (called as the global equations).**

Basic Steps

Contd.

- In this step, the algebraic equations are modified to take care of the boundary conditions on the primary variable. The modified algebraic equations are solved to find the nodal values of the primary variable.
- In the last step, the post-processing of the solution is done. That is, first the secondary variables of the problem are calculated from the solution. Then, the nodal values of the primary and secondary variables are used to construct their graphical variation over the domain either in the form of graphs (for 1-D problems) or 2-D/3-D contours as the case

may be.

Review of Matrix Algebra

Linear System of Algebraic equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned} \qquad \ldots..(1)$$

where x_1, x_2, …, x_n are the unknowns.

In *matrix form*:

$$\mathbf{Ax} = \mathbf{b} \qquad \ldots..(2)$$

$$\mathbf{Ax} = \mathbf{b} \qquad \text{.....(2)}$$

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$\mathbf{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \qquad \mathbf{b} = \{b_i\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} \qquad \text{.....(3)}$$

A is called a $n \times n$ (square) matrix, and **x** and **b** are (column) vectors of dimension n.

Row and Column Vector

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \qquad \mathbf{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

Matrix Addition and Substraction

For two matrices **A** and **B**, both of the *same size* $(m \times n)$, the addition and subtraction are defined by

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{with} \quad c_{ij} = a_{ij} + b_{ij}$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} \quad \text{with} \quad d_{ij} = a_{ij} - b_{ij}$$

Scalar Multiplication

$$\lambda \mathbf{A} = \left[\lambda a_{ij} \right]$$

Matrix Multiplication

For two matrices **A** (of size $l \times m$) and **B** (of size $m \times n$), the product of **AB** is defined by

$$\mathbf{C} = \mathbf{AB} \quad \text{with } c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

where $i = 1, 2, \ldots, l$; $j = 1, 2, \ldots, n$.

Note that, in general, $\mathbf{AB} \neq \mathbf{BA}$, but $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (associative).

Transpose of Matrix

If $\mathbf{A} = [a_{ij}]$, then the transpose of **A** is

$$\mathbf{A}^T = \left[a_{ji}\right]$$

Notice that $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$.

Symmetric Matrix

A *square* $(n \times n)$ matrix $\mathbf{A}$ is called symmetric, if

$$\mathbf{A} = \mathbf{A}^T \qquad \text{or} \qquad a_{ij} = a_{ji}$$

Unit (Identity) Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Note that $\mathbf{AI} = \mathbf{A}$, $\mathbf{Ix} = \mathbf{x}$.

Determinant of Matrix

The determinant of *square* matrix **A** is a scalar number denoted by det **A** or $|\mathbf{A}|$. For 2×2 and 3×3 matrices, their determinants are given by

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}$$

Singular Matrix

A *square* matrix $\mathbf{A}$ is *singular* if $\det \mathbf{A} = 0$, which indicates problems in the systems (nonunique solutions, degeneracy, etc.)

Matrix Inversion

For a *square* and *nonsingular* matrix $\mathbf{A}$ ($\det \mathbf{A} \neq 0$), its *inverse* $\mathbf{A}^{-1}$ is constructed in such a way that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The *cofactor matrix* $\mathbf{C}$ of matrix $\mathbf{A}$ is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant of the smaller matrix obtained by eliminating the ith row and jth column of $\mathbf{A}$.

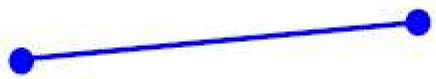
Thus, the inverse of $\mathbf{A}$ can be determined by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^T$$

We can show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

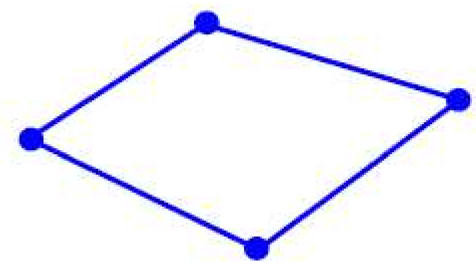
Types of Elements

1-D (Line) Element



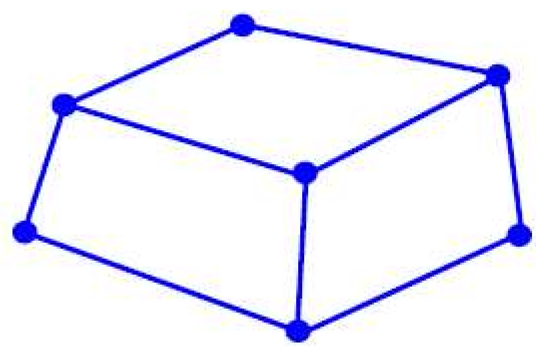
(Spring, truss, beam, pipe, etc.)

2-D (Plane) Element



mbrane, plate, shell, etc.)

3-D (Solid) Element



(3-D fields - temperature, displacement, stress, flow velocity)

The solution of the linear system of equations (Eq.(1)) can be expressed as (assuming the coefficient matrix **A** is nonsingular)

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Thus, the main task in solving a linear system of equations is to found the inverse of the coefficient matrix.

Example : Basic Concept of FEM

The most distinctive feature of the finite element method that separates it from others is the division of a given domain into a set of simple subdomains, called *finite elements*. Any geometric shape that allows computation of the solution or its approximation, or provides necessary relations among the values of the solution at selected points, called *nodes*, of the subdomain, qualifies as a finite element. Other features of the method include seeking continuous, often polynomial, approximations of the solution over each element in terms of nodal values, and assembly of element equations by imposing the interelement continuity of the solution and balance of interelement forces. Here the basic ideas underlying the finite element method are introduced via two simple examples:

1. Determination of the circumference of a circle using a finite number of line segments

Summary

References

- **Finite Element Method – Y. M. Desai, T. I. Eldho and A. H. Shah (Pearson)**
- **An Introduction of Finite Element Method- J. N. Reddy , McGraw – Hill**
- **Introduction of Finite Element Method- Yijun Liu**


THANK YOU

Finite Difference Method

Lecture03

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Introduction

- Many physical phenomenon in applied science and engineering when formulated into mathematical models fall into a category of systems known as differential equations.
- The basis of the finite difference method is to replace the differential equation and expression defining the boundary conditions (if they contain derivatives of the function) by corresponding difference equation. This way, the solution of differential equation is reduced to one of solving set of algebraic equations.

PARTIAL DIFFERENTIAL EQUATION

- Physical phenomena in applied science and engineering when formulated into mathematical models fall into a category of systems known as partial differential equations that involves more than one independent variable which determine the behaviour of the dependent variable as described by their partial derivative contained in the equation.

Examples: Heat flow in a rectangular plate

The model for heat flow in a rectangular plate that is heated is given by:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Where u(x, y) denotes the temperature at point (x, y) and f(x, y) is the heat source. Here, the rate of change of a variable is expressed as a function of variables and parameters. Although most of the differential equations may be solved analytically in their simplest form, analytical techniques fail when the models are modified to take into account the effect of other conditions of real life situations.

- If we represent the dependent variable as 'f' and the two independent variables as 'x' and 'y', then we will have three possible second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\,\partial y} \text{ and } \frac{\partial^2 f}{\partial y^2}$$

Then we can write a second order equation involving two independent variables in general form as:

$$a\frac{\partial^2 f}{\partial x^2} + b\,\frac{\partial^2 f}{\partial x\,\partial y} + c\frac{\partial^2 f}{\partial y^2} = F\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \qquad \text{....(A)}$$

Where the coefficients a, b and c may be constants or function of 'x' and 'y'. Depending on the values of these coefficients equation (A) can be classified into one of the three types of equations, namely:

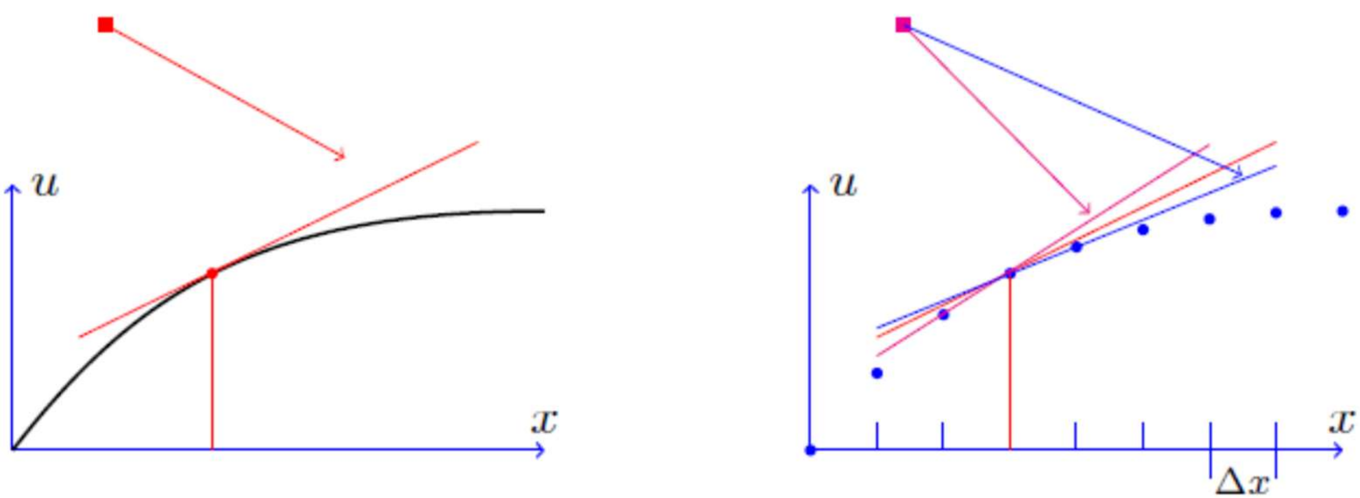
a) Elliptic Equation: If $b^2 - 4ac < 0$
b) Parabolic Equation: If $b^2 - 4ac = 0$
c) Hyperbolic Equation: If $b^2 - 4ac > 0$

Finite Difference Method

- In finite difference method, replacing the derivatives that occur in the DE by their difference equivalents.
- FDM based on formulae for approximating the first and second order derivatives of a function.

Finite differences: natural approximations to **derivatives**

$$\frac{du(x)}{dx} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \Leftrightarrow \frac{\Delta u(x)}{\Delta x} \equiv \frac{u(x+\Delta x) - u(x)}{\Delta x}$$



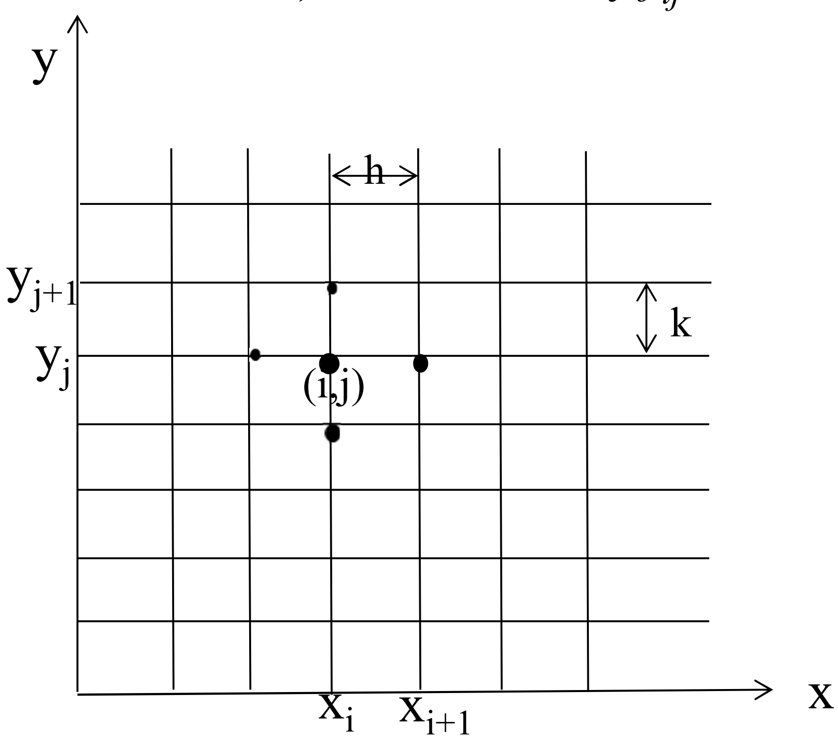
Discrete step Δx can be small but **it does not tend to zero!**

Finite Difference Schemes

Forward difference	$\frac{u(x+h)-u(x)}{h}$
Backward difference	$\frac{u(x)-u(x-h)}{h}$
Centred difference	$\frac{u(x+h)-u(x-h)}{2h}$

Deriving Difference Equations

- Solution domain for two dimensional problem is split into two rectangular Grids of width h and height k.
- The pivotal values at the point of intersection (known as grid points or nodes) are defined by f_{ij} which is a function of two space variable x and y



Two dimensional finite difference grid

$$x_{i+1} = x_i + h$$
$$y_{j+1} = y_j + k$$

Central Difference Scheme

- First and Second derivatives of function f(x) can be obtained by using Taylor's expansion series .

Taylor's Expansion series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots$$
$$+ higher\ \ order\, terms --(1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \ldots\ldots.$$
$$+ higher\ \ order\, terms --(2)$$

Contd.

Taylor's Expansion series

- **First order derivative**

Subtracting (2) to (1) results

$$f(x+h)-f(x-h)=2hf'(x)+\frac{2h^3}{3!}f'''(x)$$

$$(neglecting\,it)$$

$$f'(x)=\frac{f(x+h)-f(x-h)}{2h}\,.................(3)$$

- **Second order derivative**

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ...(4)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + ...(5)$$

Adding (4) to (5) results

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + ...$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \quad ....(6)$$

- If the function f(x) has continuous fourth derivative, then its first and second derivatives are given by the following central difference approximations.

$$f'(x_i) = \frac{f(x_i+h)-f(x_i-h)}{2h} = \frac{f_{i+1}-f_{i-1}}{2h}$$

$$f''(x_i) = \frac{f(x_i+h)-2f(x_i)+f(x_i-h)}{h^2} = \frac{f_{i+1}-2f_i+f_{i-1}}{h^2}$$

The subscript on f on indicates the x-value at which the function is evaluated.

- **When ‘f’ is a function of two variables ‘x’ and ‘y’, the partial derivatives of ‘f’ with respect to ‘x’ (or ‘y’) are the ordinary derivatives of ‘f’ with respect to ‘x’ (or ‘y’) when ‘y’ (or ‘x’) does not change. So, we can use above equation to determine derivatives with respect to ‘x’ and in the ‘y’ direction.**

$$\frac{\partial f(x_i,y_j)}{\partial x} = f_x(x_i, y_j) = \frac{f(x_{i+1},y_j)-f(x_{i-1},y_j)}{2h}$$

$$\frac{\partial f(x_i,y_j)}{\partial y} = f_y(x_i, y_j) = \frac{f(x_i,y_{j+1})-f(x_i,y_{j-1})}{2k}$$

$$\frac{\partial^2 f(x_i,y_j)}{\partial x^2} = f_{xx}(x_i, y_j) = \frac{f(x_{i+1},y_j)-2f(x_i,y_j)+f(x_{i-1},y_j)}{h^2}$$

$$\frac{\partial^2 f(x_i,y_j)}{\partial y^2} = f_{yy}(x_i, y_j) = \frac{f(x_i,y_{j+1})-2f(x_i,y_j)+f(x_i,y_{j-1})}{k^2}$$

$$\frac{\partial^2 f(x_i,y_j)}{\partial x\,\partial y} = f_{xy}(x_i, y_j) = \frac{f(x_{i+1},y_{j+1})-f(x_{i+1},y_{j-1})-f(x_{i-1},y_{j+1})-f(x_{i-1},y_{j-1})}{4hk}$$

References

- **Numerical methods – E. Balagurswamy, Tata McGraw – Hill**
- **An Introduction of Finite Element Method- J. N. Reddy , McGraw – Hill**
- **Finite Element Method – Y. M. Desai**


THANK YOU

Finite Difference Method- Solution of PDE

Lecture04

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Solution of Partial Differential Equation

- Elliptic equations are governed by conditions on the boundary of closed domain. We consider here the two most commonly encountered elliptic equations, namely: Laplace's Equation and Poisson's Equation.

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x\,\partial y} + c\frac{\partial^2 f}{\partial y^2} = F(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \qquad \ldots\text{(A)}$$

Laplace Equation

The general second order partial differential equation (A) in previous lecture, when a = 1, b = 0, c = 1 and F(x, y, fx, fy) = 0 becomes:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f = 0\,. \qquad \left\{\because\ \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}$$

The operator ∇^2 is called the Laplacian Operator and equation is called Laplace's Equation.

To solve the Laplace's Equation on a region in the xy-plane, we subdivide the region as shown in the figure below. Consider the portion of the region near (**x_i, y_i**). We have to approximate:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f = 0$$

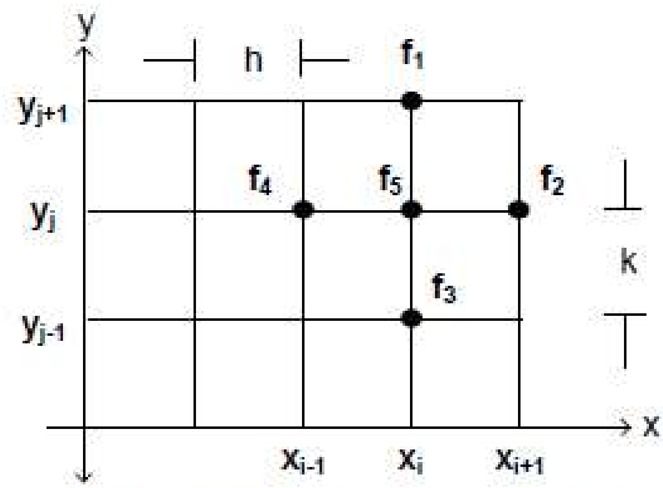


Fig: Solution of Laplace's Equation

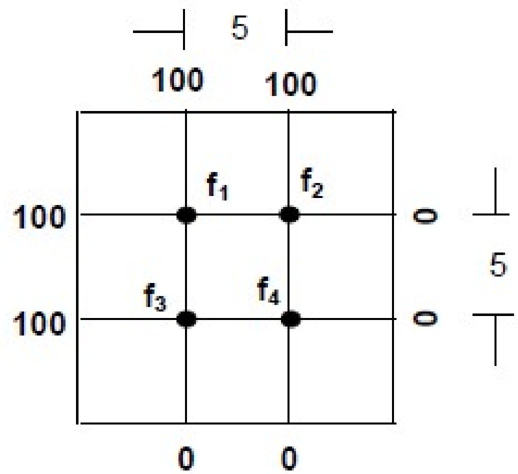
Replacing the second order derivatives by their finite difference equivalents at point (**x_i, y_i**). we obtain.

$$\frac{f_2 - 2f_5 + f_4}{h^2} + \frac{f_1 - 2f_5 + f_3}{k^2} = 0$$

i.e. $f_2 + f_4 + f_1 + f_3 - 4f_5 = 0$ (for $h = k$)

Example:

Consider a steel plate of size 15X15 cm. If two of the sides are held at 100C and the other two sides are held at 0C. What is the steady state value of temperature at interior knots, assuming a grid size of 5X5 cm.



Solution

At point 1:

$f_2 + f_3 + 100 + 100 - 4f_1 = 0 \Longrightarrow f_2 + f_3 - 4f_1 + 200 = 0$ ... (i)

$f_2 + f_3 - 4f_1 + 200 = 0$ ... (i)

At point 2:

$f_1 + f_4 + 100 + 0 - 4f_2 = 0 \Rightarrow f_1 - 4f_2 + f_4 + 100 = 0$ ... (ii)

At point 3:

$f_1 + f_4 + 100 + 0 - 4f_3 = 0 \Rightarrow f_1 - 4f_3 + f_4 + 100 = 0$ ... (iii)

At point 4:

$f_3 + 0 + 0 + f_2 - 4f_4 = 0 \Rightarrow f_2 + f_3 - 4f_4 = 0 \Rightarrow f_2 = 4f_4 - f_3$ ... (iv)

Putting the values of of f_2 from equation (iv) into equations (i) and (ii), we get:

$f_2 + f_3 - 4f_1 + 200 = 0 \gg 4f_4 - f_3 + f_3 - 4f_1 + 200 = 0 \gg f_1 - f_4 = 50$ ... (vi)
$f_1 - 4f_2 + f_4 + 100 = 0 \gg f_1 - 4(4f_4 - f_3) + f_4 + 100 = 0 \gg f_1 - 4f_3 - 15f_4 = -100$ ... (vii)
$f_1 - 4f_3 + f_4 + 100 = 0$ ... (viii)

Now, solving equations (vi), (vii) and (viii) we will get: $f_1 = 75$, $f_3 = 50$ and $f_4 = 25$
Putting f_3 and f_4 in equation (iv), we get: $f_2 = 50$.

Poisson Equation

The general second order partial differential equation, when a = 1, b = 0, c = 1 and F(x, y, fx, fy) = g(x, y) becomes:

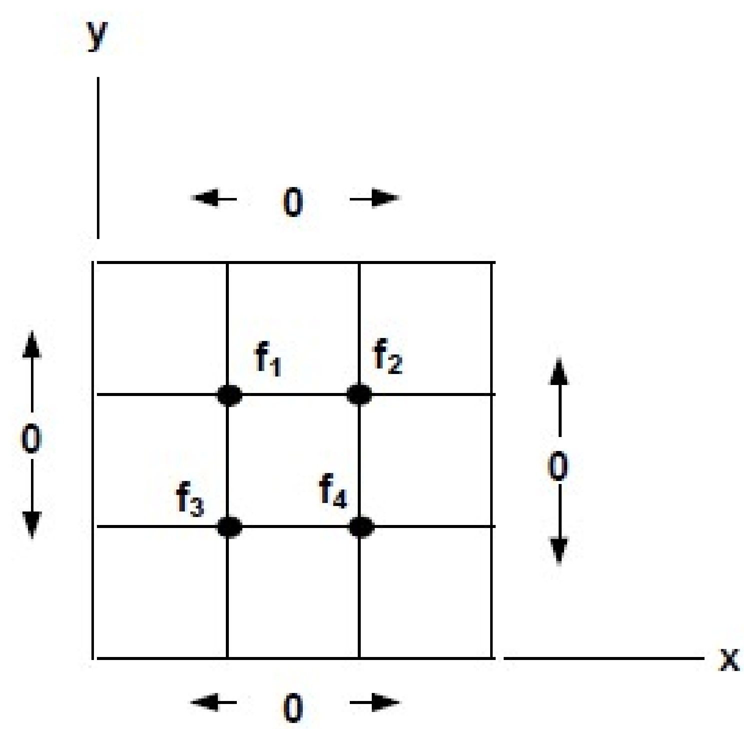
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f = g(x, y) \qquad \left\{\therefore\ \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}$$

This equation is called Poisson's Equation. Now, Laplace's equation may be modified to solve Poisson's Equation. Then the finite difference formula for solving Poisson's Equation takes the form:

i.e. $f_2 + f_4 + f_1 + f_3 - 4f_5 = h^2 g_{ij}$ (for h = k)

Example

Solve the Poisson's Equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \leq x \leq 3$ and $0 \leq y \leq 3$ with f = 0 on the boundary and h = 1.



Solution

At point 1: $0 + 0 + f_2 + f_3 - 4f_1 = 2(1)^2(2)^2$

At point 2: $0 + 0 + f_1 + f_4 - 4f_2 = 2(2)^2(2)^2$

At point 3: $0 + 0 + f_1 + f_4 - 4f_3 = 2(1)^2(1)^2$

At point 4: $0 + 0 + f_2 + f_3 - 4f_4 = 2(2)^2(1)^2$

$f_2 + f_3 - 4f_1 = 8$ ... (i)

$f_1 - 4f_2 + f_4 = 32$ ... (ii)

$f_1 - 4f_3 + f_4 = 2$ ... (iii)

$f_2 + f_3 - 4f_4 = 8$ ... (iv)

On solving these simultaneous equations by elimination method, we will get the answers.

$f_1 = -22/4 \quad f_2 = -43/4$

$f_3 = -13/4 \quad f_4 = -22/4$

Comparison of FDM to FEM

FDM is subset of FEM. Both FEM and FDM dicretize a continuum and both generate simultaneous algebraic equations to be solved for nodal degree of freedom.

•FDM makes pointwise approximations to the governing equations ie. it ensures continuity only at the node points. Continuity along the sides of grid line are not ensured. FEM makes piecewise approximation ie.it ensures the continuity at node points as well as along the sides of element.

•FDM do not give the values at any point except at node points. It do not give any approximating functions to evaluate the basic values (deflections ,in case of solid mechanics) using the nodal value. FEM can give the values at any points other than nodes are by using suitable interpolation formlae.

•FDM makes stair type of approximation to sloping and curved boundary. FEM can consider the sloping boundaries exactly. If curved elements are used, even the curved boundaries can be handled exactly.

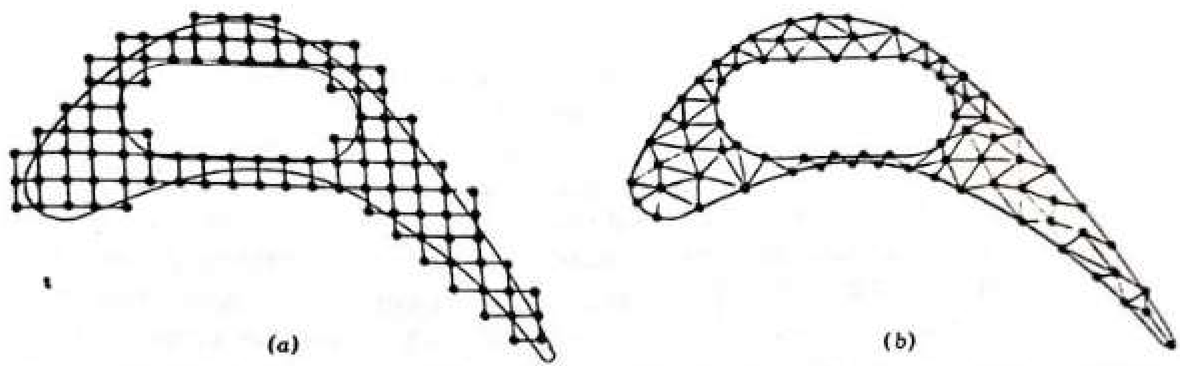


Fig. FEM and FDM discritization of turbine blade profile (a) typical FEM discritization (b) typical FDM discritization

•FDM needs larger number of nodes to get good results while FEM needs fewer nodes.
•With FDM fairly complicated problems can be handled where as FEM can handle all complicated problems.
•FEM can be easily coupled with CAD model.

Applications

- FEM is now applicable to a wide range of engineering problems.
- Majority of applications of FEM are in the area of solid mechanics and fluid mechanics.
- In the last two decades, FEM has also been used in electrical and electromagnetic problems as well as in bioengineering problems.
- Categories of problems that can be solved using FEM can be divided into equilibrium, eigenvalue and transient problems.
- The equilibrium problems are generally steady-state problems such as determination of stress and displacement in solid mechanics-related problems, determination of temperature distribution in thermal problems, estimation of potential, velocity and pressure in fluid mechanics problems.

- The eigenvalue problems are also steady state in nature but include estimation of vibration and natural frequencies in solids and fluids.
- In the transient problems, FEM is used in propagation problems of continuum mechanics with respect to time.

Applications of FEM in various engineering fields

Structural mechanics and aerospace engineering: FEM applications in equilibrium conditions include analysis of beams, plates, shell structures, stress and torsion analysis of various structures. On the other hand, eigenvalue analyses include stability of structures, viscoelastic damping, vibrations and natural frequency analysis of structures. The transient or propagation analysis using FEM includes dynamic response of structures to periodic loads, viscoelastic and thermo-elastic problems and stress wave propagation.

Geotechnical engineering: FEM applications include stress analysis, slope stability analysis, soil structure interactions, seepage of fluids in soils and rocks, analysis of dams, tunnels, bore holes, propagation of stress waves and dynamic soil structure interaction.

Fluid mechanics, hydraulic and water resources engineering: FEM applications include solutions of potential and viscous flow of fluids, steady and transient seepage in aquifers and porous media, movement of fluids in containers, external and internal flow analysis, seiche of lakes, ocean and harbors, salinity and pollution studies in surface and sub-surface water problems, sediment transport analysis and water distribution networks.

Mechanical engineering: In mechanical engineering, FEM applications include steady and transient thermal analysis in solids and fluids, stress analysis in solids, automotive design and analysis and manufacturing process simulation.

Nuclear engineering: FEM applications include steady and dynamic analysis of reactor containment structures, thermo-viscoelastic analysis of reactor components, steady and transient temperature-distribution analysis of reactors and related structures

Electrical and electronics engineering: FEM applications include electrical network analysis, electromagnetics, insulation design analysis in high-voltage equipment's, thermosonic wire bond analysis, dynamic analysis of motors, molding process analysis in encapsulation of integrated circuits and heat analysis in electrical and electronic equipment.

Metallurgical, chemical and environmental engineering: In metallurgical engineering, FEM is used for the metallurgical process simulation, molding and casting. In chemical engineering, FEM can be used in the simulation of chemical processes, transport processes (including advection and diffusion) and chemical reaction simulations. FEM is used in environmental engineering widely in the areas of surface and sub-surface pollutant transport modelling, air pollution modelling, land-fill analysis and environmental process simulation.

Meteorology and bioengineering: In the recent times, FEM is used in climate predictions, monsoon prediction and wind predictions. FEM is also used in bioengineering for the simulation of various human organs, blood circulation prediction and even total synthesis of human body. e.g. Dentist can analyze the various aspects of teeth.

Merits and Demerits of FEM

FEM can be applied to almost all branches of engineering. The fact that FEM can be used to solve a particular problem does not mean that it is the most ideal solution technique. To solve a given problem, often several attractive numerical techniques are available. Each method has its own merits and demerits. Depending on the problem, ‘the best’ method should be chosen by comparing the merits and demerits of the method.

Merits

Compared to other numerical methods some of the merits of FEM are as follows.

Modelling of complex geometries and irregular shapes are easier as varieties of finite elements are available for discretization of domain.

Boundary conditions can be easily incorporated in FEM.

Different types of material properties can be easily accommodated in modelling from element to element or even within an element. Higher order of elements may be implemented with relatively ease.

Problems with heterogeneity, anisotropy, nonlinearity and time-dependency can be easily dealt with.

The systematic generality of FEM procedure makes it a powerful and versatile tool for a wide range of problems.

FEM is simple, compact and result-oriented and hence widely popular among engineering community.

FEM can be easily coupled with computer-aided design (CAD) programs in various streams of engineering.

An FEM model can be developed at different levels and it is possible to interpret the method in physical terms.

In FEM, it is relatively easy to control the accuracy by refining the mesh or using higher order elements.
Availability of large number of computer software packages and literature makes FEM a versatile and powerful numerical method.

Demerits

Numerical solution is obtained at one time for a specific problem case only. Hence, unlike analytical solutions, there is no advantage of flexibility and generalization.
Large amount of data is required as input for the mesh used in terms of nodal connectivity and other parameters depending on the problem.
Generally, voluminous output data must be analyzed and interpreted. Requires digital computer and extensive software.
Experience, good engineering judgment and understanding of the physical problems are required in FEM modelling.
Poor selection of element type or discretization may lead to faulty results.

Summary

References

- **Numerical Methods – E. Balagurswamy, Tata McGraw – Hill**
- **The Finite Element Method for Engineers- K H Huebner, John Wiley & Sons**
- **Finite Element Method- Bhavikatti**


THANK YOU

METHOD OF WEIGHTED RESIDUALS

Lecture05

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METHOD OF WEIGHTED RESIDUALS

- The MWR is a very useful tool to approximate solutions to differential equations.
- In MWR, the differential equation is considered directly rather than its variational forms.
- An approximate solution has some error with respect to the exact solution. Depending on the problem, error may be in the domain or on the boundary or in both. The objective of the method is to reduce the error.
- The procedure through which the error minimization is implemented is called the weighted residual technique. When method of weighted residual is used, the concerned problem is represented by differential equation valid over the domain Ω with the prescribed boundary conditions on the boundary Γ.

Contd.

METHOD OF WEIGHTED RESIDUALS

- Initially, a trial function is assumed as solution. The function may not satisfy the differential equation and the boundary conditions exactly. By substituting the trial function in the differential equation and boundary condition, an error known as residual will be produced.
- The unknown parameters in the trial function are determined in such a way that the residual is made as small as possible.
- Some of the most commonly used weighted residual techniques in engineering applications are :

(i)Method of Point Collocation

(ii) Method of Least Squares, and

(iii)Galerkin's Approach.

Method of Point Collocation

- A number of discrete points referred to as collocation points are chosen in the collocation method and the error is forced to become zero at these points. As a result, the differential equation is satisfied at these chosen points.
- Let $L\phi = 0$ be the differential equation under consideration. Approximate solution should be such that the polynomial or any other function chosen should satisfy the boundary conditions of the problem, i.e., ϕ can be expressed by

$$\phi = \sum_{i=1}^{n} N_i \phi_i \quad \text{..................(1)}$$

Method of Point Collocation

- ϕ_i are undetermined parameters and N_i are known functions linearly independent in dimensionless forms. Parameters ϕ_i are determined by enforcing the error or residual by $\varepsilon\{=L(\phi)\} = 0$ at n points in the domain resulting in n equations to determine n values of ϕ_i.

- The number of collocation points depends on the number of unknowns in the assumed function.

Method of Point Collocation

- Depending upon the problem, the point collocation technique can be classified into three methods.
- In interior collocation method, the admissible function satisfies all the boundary conditions but not the differential equation in the domain. Here, errors arising are set to zero at chosen points in the interior of the domain.
- In boundary collocation method, the admissible function satisfies the differential equation exactly in the domain, but does not satisfy the boundary conditions. Here the collocation points are chosen at the boundary.
- In mixed collocation method, the admissible function does not satisfy the differential equation and boundary conditions exactly. Thus, collocation points are chosen in the domain as well as on the boundaries.

Example on Point Collocation Method

- Solve the differential equation using boundary conditions $\phi(x = 0) = 0$ and $\phi(x = 1) = 0$.

$$\frac{d^2\phi}{dx^2} - \phi = x \quad \text{........(1)}$$

Choose x = 0.25 and x = 0.5 as collocation points.

Solution:

The exact solution for the expression (1)

$$\phi = 0.8509 \sinh x - x \quad \text{.........(2)}$$

- Let the approximate solution that satisfies the boundary conditions be of the form

$$\phi = x(1-x)(\alpha_1 + \alpha_2 x.......) \quad .....(3)$$

Or

$$\phi = \alpha_1(x - x^2) + \alpha_2(x^2 - x^3) + ...(4)$$

$$\left(\quad \phi = \sum_{i=1}^{n} N_i \phi_i \right.$$

Considering only the first two coefficients,

$$\frac{d\phi}{dx} = \alpha_1 + 2x(\alpha_2 - \alpha_1) - 3\alpha_2 x^2 \quad .....(5)$$

$$\frac{d^2\phi}{dx^2} = 2(\alpha_2 - \alpha_1) - 6\alpha_2 x \quad .....(6)$$

Substitute (4) and (6) into eqn (1) yield an error

$$\in = \in(x) = \alpha_1(x^2 - x - 2) - \alpha_2(6x + x^2 - x^3 - 2) - x \quad .....(7)$$

Choosing x=0.25 and x=0.5 as collocation points, where the error is made equal to zero.

$$\in(x = 0.25) = -140\alpha_1 + 29\alpha_2 - 16 = 0 \quad ......(8)$$

$$\in(x = 0.5) = -18\alpha_1 + 9\alpha_2 - 4 = 0 \quad ....(9)$$

Solving eqn(8) and (9) simultaneously,

$$\alpha_1 = -0.1459, \qquad \alpha_2 = -0.1526$$

Hence the solution

$$\phi = x(x-1)(0.1459 + 0.1526x) \qquad ....(10)$$

Number of collocation points should be equal to number of unknown parameters α .

References

- **Finite element Method- Y. M. Desai, T. I. Eldho & A. H. Shah, Pearson**
- **An Introduction of Finite Element Method- J. N. Reddy , McGraw – Hill**


THANK YOU

METHOD OF WEIGHTED RESIDUALS

Lecture06

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METHOD OF WEIGHTED RESIDUALS (Contd.)

❖ Method of Least Square

- In the method of least squares, the sum of squares of the residuals are minimal (or made zero) after substitution of the approximate solution in the differential equation. That is, the sum of the squares of the error is minimized as

$$F = \sum \varepsilon^2 = \sum \{L(\phi)\}^2$$

For F to be minimum $\frac{\partial F}{\partial \phi} = 0, i = 1, \ldots n$ that yields n equations

to determine n values of ϕi.
Thus, the final equations are

$$\int L(\phi) \frac{\partial L}{\partial \phi} d\Omega = 0$$

Where $\frac{\partial L}{\partial \phi}$ are the weights

Example on Least Squares Method

- Solve the differential equation using boundary conditions $\phi(x = 0) = 0$ and $\phi(x = 1) = 0$.

$$\frac{d^2\phi}{dx^2} - \phi = x$$

Solution:

The exact solution for the expression

$$\phi = 0.8509 \sinh x - x \quad ....$$

- **Let the approximate solution that satisfies the boundary conditions be of the form**

$$\phi = x(1-x)(\alpha_1 + \alpha_2 x.......) \quad ....$$

Or

$$\phi = \alpha_1(x - x^2) + \alpha_2(x^2 - x^3) + ...$$

$$\frac{d\phi}{dx} = \alpha_1 + 2x(\alpha_2 - \alpha_1) - 3\alpha_2 x^2 \quad ...$$

$$\frac{d^2\phi}{dx^2} = 2(\alpha_2 - \alpha_1) - 6\alpha_2 x \quad ....$$

Error is given by

$$\varepsilon = \alpha_1(x^2 - x - 2) - \alpha_2(-x^3 + x^2 + 6x - 2) - x$$

Then,

$$\frac{\partial}{\partial \alpha_1}(\varepsilon^2) = [\alpha_1(x^2 - x - 2) - \alpha_2(6x + x^2 - x^3 - 2) - x]2(x^2 - x - 2)$$

and

$$\frac{\partial}{\partial \alpha_2}(\varepsilon^2) = [\alpha_1(x^2 - x - 2) - \alpha_2(6x + x^2 - x^3 - 2) - x](6x + x^2 - x^3 - 2) \times 2$$

Hence, $$\int_0^1 [\alpha_1(x^2 - x - 2) - \alpha_2(6x + x^2 - x^3 - 2) - x](x^2 - x - 2)\, dx = 0$$

and $$\int_0^1 [\alpha_1(x^2 - x - 2) - \alpha_2(6x + x^2 - x^3 - 2) - x](6x + x^2 - x^3 - 2)\, dx = 0$$

Simplifying Eqs. $$\int_0^1 [\alpha_1(x^2 - x - 2) - \alpha_2(6x + x^2 - x^3 - 2) - x](x^2 - x - 2)\,dx = 0$$
$$\int_0^1 [\alpha_1(x^2 - x - 2) - \alpha_2(6x + x^2 - x^3 - 2) - x](6x + x^2 - x^3 - 2)\,dx = 0$$

resulting equations are

$$4.67\alpha_1 + 2.35\alpha_2 = -1.084$$
$$-2.35\alpha_1 - 4.276\alpha_2 = 1.05$$

Solving,

$$\alpha_1 = -0.1500 \text{ and } \alpha_2 = -0.16309$$

Hence, $$\phi = x(x-1)(0.1500 + 0.16309x)$$

Galerkin's Method

- In the Galerkin's method, approximating or trial functions are considered to be the weighting functions. Consider a system with homogeneous boundary conditions

$$L(\phi) = g \quad \text{in } \Omega \qquad \text{-----(A)}$$

- An approximate function that satisfies these conditions can be used such that

$$\phi \approx \bar{\phi} = \sum_{i=1}^{n} N_i \Psi_i$$

- Here *Ni* are the assumed functions and *Ψi* are either the unknown parameters or unknown functions of one of the independent variables. Substituting this approximating function in Eq. (A) may produce a residual.

Residual

$$R = L(\overline{\phi}) - g \neq 0$$

- In the Galerkin's method, the error is made orthogonal to the same trial function *Ψi* i.e.,

$$\int R\, w_i \,\mathrm{d}\,\Omega = 0, \quad \text{where } w_i = N_i$$

- This method also yields *n* linear equations to determine *n* values of *Ψi*. Galerkin's method may produce symmetrical coefficients in many cases since the weighting function and approximating functions are same.
- Many finite element formulations are based on the Galerkin's type approximating method.

Example on Galerkin's Method

- Solve the differential equation using boundary conditions ϕ(x = 0) = 0 and ϕ(x = 1) = 0.

$$\frac{d^2\phi}{dx^2} - \phi = x \qquad \text{........}(1)$$

Solution:

The exact solution for the expression (1)

$$\phi = 0.8509 \sinh x - x \qquad \text{.........}(2)$$

- Let the approximate solution that satisfies the BCs be of the form

$$\phi = x(1-x)(\alpha_1 + \alpha_2 x.......) \quad .....(3)$$

$$\phi = \alpha_1(x - x^2) + \alpha_2(x^2 - x^3) + ...(4)$$

- The functional form of error is given by

$$N_1 = (x - x^2) \text{ and } N_2 = (x^2 - x^3)$$

That is,

$$\int_0^1 \varepsilon(x - x^2)\,dx = 0 \quad ....(5)$$

and

$$\int_0^1 \varepsilon(x^2 - x^3)\,dx = 0 \quad ....(6)$$

- Simplifying equations (5) and (6), resulting equations are

$$-0.36667\alpha_1 - 0.18333\alpha_2 = 0.083333 \quad \text{....(7)}$$
$$-0.18333\alpha_1 - 0.14286\alpha_2 = 0.05 \quad \text{....(8)}$$

Solving equations (7) and (8) yields $\alpha 1 = -0.14588$ and $\alpha 2 = -0.16278$

$$\phi = x(x-1)(0.14588 + 0.16278x)$$

Comparison of Results Obtained by Different Methods

		Point Collocation		Least Square		Galerkin's Method	
x	Exact ϕ	ϕ	% Error	ϕ	% Error	ϕ	% Error
0.25	−0.03505	−0.03451	1.54	−0.03576	2.02	−0.034976	0.211
0.50	−0.0566	−0.055554	1.84	−0.057886	2.27	−0.056825	0.397
0.75	−0.0503	−0.04882	2.94	−0.051059	1.50	−0.050240	0.112

The above comparison shows that Galerkin's method yields better results in comparison with the Point collocation or least square method.

References

- **Finite element Method- Y. M. Desai, T. I. Eldho & A. H. Shah, Pearson**
- **An Introduction of Finite Element Method- J. N. Reddy , McGraw – Hill**
- **Applied Finite element analysis for engineers- Frank L. Stasa, CBS International**


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Rayleigh–Ritz Method
Lecture07

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in the Rayleigh–Ritz Method. Thus,.

Rayleigh–Ritz Method

- In Rayleigh–Ritz Method, Form of the unknown solution is assumed in terms of known functions (trial functions) with unknown adjustable parameters.
- From the family of trial functions, the function that renders the functional stationary are selected and substituted into the functional, which is a function of the functions.
- The functional is expressed in terms of the adjustable parameters

Rayleigh–Ritz Method

- The resulting functional is differentiated with respect to each parameter and the resulting equation is set equal to zero.
- If there are *n* unknown parameters in the functional, there will be *n* simultaneous equations to be solved for the parameters.
- The best solution is obtained from the family of assumed solutions.
- Accuracy depends on the trial functions chosen. Generally, the trial function is constructed from the polynomials of successively increasing degree.

- The main aim of Rayleigh–Ritz method is to replace the problem of finding the minima and maxima of integrals by finding the minima and maxima of functions of several variables.
- For example, consider search of a function $L(x)$ that will extremize certain given functional (L). As mentioned, $L(x)$ can be approximated by a linear combination of suitably chosen coordinate functions $c1(x)$, $c2(x)$,………$cn(x)$. Then $L(x)$ can be written as

$$\frac{\partial I}{\partial g_j} = 0 \quad (j = 1, 2, 3, \ldots n) \qquad (2.36)$$

- n algebraic equations are obtained from which the unknown constants g_j are determined.

- Solve the following differential equation using Rayleigh–Ritz Method. Use boundary conditions ϕ(x = 0) = 0 and ϕ(x = 1) = 0.

$$\frac{d^2\phi}{dx^2} - \phi = x \qquad \text{-------(1)}$$

Solution

- Solution is obtained from the corresponding variational functions. Variational function for the above differential equation can be written as

$$\psi = I(\phi) = \int F(x,\phi,\phi')dx \qquad \text{-------(2)}$$

$$\text{where } F(x, \phi, \phi') = \left(\frac{d\phi}{dx}\right)^2 + \phi^2 + 2x\phi$$

- *F* satisfies the Euler–Lagrangian equation of variational calculus given as

$$\frac{\partial F}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \phi'}\right) = 0 \qquad \text{-------(3)}$$

yielding the differential equation chosen.

$$\frac{\partial F}{\partial \phi} = 2\phi + 2x, \quad \frac{\partial F}{\partial \phi'} = 2\phi', \quad \frac{d}{dx}\left(\frac{\partial F}{\partial \phi'}\right) = 2\frac{d^2\phi}{dx^2}$$

substituting all these into Eq. (3)

$$2\phi + 2x - 2\frac{d^2\phi}{dx^2} = 0 \text{ or } \phi'' - \phi = x$$

→ **It is given differential equation.**

In the Euler–Lagrange equation, it is to find the function (the Lagrangian) that makes the action (a functional) stationary, which is in practice done by solving a differential equation

In the **calculus** of variations, the **Euler equation** is a second-order partial differential **equation** whose solutions are the functions for which a given functional is stationary.

A stationary point of a function f(x) is a point where the derivative of f(x) is equal to 0.

- Let the approximate solution that satisfies the BCs be of the form

$$\phi = x(1-x)(\alpha_1 + \alpha_2 x \ldots\ldots)$$

$$\phi = \alpha_1(x - x^2) + \alpha_2(x^2 - x^3) + \ldots.(4)$$

$$\frac{d\phi}{dx} = \alpha_1 + 2x(\alpha_2 - \alpha_1) - 3\alpha_2 x^2 \quad \ldots..(5)$$

- Substituting eqn(4) and eqn(5) into (6) and integrating separately term by term within the limits $x = 0$ to 1,

$$\psi = \int\left[\left(\frac{d\phi}{dx}\right)^2 + \phi^2 + 2x\phi\right]dx \quad \text{-------}(6)$$

$$\int_0^1 \left(\frac{d\phi}{dx}\right)^2 = \int_0^1 (\alpha_1 + 2x(\alpha_2 - \alpha_1) - 3\alpha_2 x^2)^2 \, dx = \frac{\alpha_1{}^2}{3} + \frac{2}{15}\alpha_2{}^2 + \frac{\alpha_1 \alpha_2}{3}$$

$$\int_0^1 \phi^2 dx = \frac{\alpha_1^2}{30} + \frac{\alpha_2^2}{105} + \frac{\alpha_1 \alpha_2}{30}$$

$$\int_0^1 2x\, \phi dx = \frac{\alpha_1}{6} + \frac{\alpha_2}{10}$$

- **The variational function can be written as**

$$\psi = \frac{11}{30}\alpha_1^2 + \frac{1}{7}\alpha_2^2 + \frac{11}{30}\alpha_1 \alpha_2 + \frac{1}{6}\alpha_1 + \frac{1}{10}\alpha_2 \quad \text{-------}(7)$$

- Minimizing Ψ with respect to undetermined coefficients α1, and α2

$$\frac{\partial \psi}{\partial \alpha_1} = \frac{11}{15}\alpha_1 + \frac{11}{30}\alpha_2 + \frac{1}{6} = 0 \qquad \text{-------(8)}$$

$$\frac{\partial \psi}{\partial \alpha_2} = \frac{11}{30}\alpha_1 + \frac{2}{7}\alpha_2 + \frac{1}{10} = 0 \qquad \text{-------(9)}$$

- Solving eqn(8) and eqn(9),simultaneously, $\alpha_1 = -0.1460$ and $\alpha_2 = -0.1626$

- Hence expression is

$$\phi = x(x-1)\,(0.1460 + 0.1626x)$$

Relation Between FEM and Rayleigh–Ritz Method

- From the mathematical point of view, the FEM can be considered as a special case of Rayleigh–Ritz Method.
- Both FEM and Rayleigh–Ritz Method use a set of trial functions as the starting point for obtaining the approximate solution, take linear combinations of trial functions and try to get combinations of trial functions, which make a functional stationary.
- The trial function is considered for the entire domain in the Rayleigh–Ritz Method and hence simple domain only can be modeled. On the other hand, functions are not defined over the whole solution domain in FEM.
- The trial function is defined element wise and they do not have to satisfy boundary conditions.

References

- **Finite element Method- Y. M. Desai, T. I. Eldho & A. H. Shah, Pearson**
- **An Introduction of Finite Element Method- J. N. Reddy , McGraw – Hill**
- **Applied Finite element analysis for engineers- Frank L. Stasa, CBS International**


THANK YOU

Direct Approach–Finite Element Method
Lecture08

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Different Approaches used in FEM

- There are number of ways in which one can formulate the properties of individual elements of the domain. Most commonly used approaches to formulate element matrices are:

(i) direct approach

(ii) variational approach

(iii) energy approach

(iv) weighted residual approach.

Direct Approach

- The basic idea of finite element method was conceived from the physical procedure used in framed structural analysis and network analysis (pipe network and electric network).
- It may be possible to choose elements in a way that leads to an exact representation of the problem in certain applications.
- In the direct approach, element properties are derived from the fundamental physics and nature of the problem.

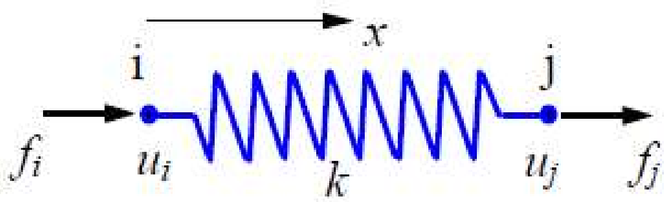
Direct Approach

Contd.

- Main advantage of this approach is that an easy understanding of techniques and essential concepts is gained without much mathematical illustrations.
- In solving a problem using the direct approach, first, the elements are defined and then their properties are determined. Once the elements have been selected, direct physical relationships are used to establish element equations in terms of concerned variables.
- Finally, element equations for various elements or members are combined to generate a system of equations which are solved for the unknowns.
- Some engineering problems which can be solved using the direct approach include spring systems, trusses, beams, fluid flow in pipe networks, electric resistance networks etc.
- The linear spring system problem is analyzed to demonstrate the development of finite element technique using the direct approach formulation.

Spring

One Spring Element



Two nodes: i, j

Nodal displacement: u_i , u_j

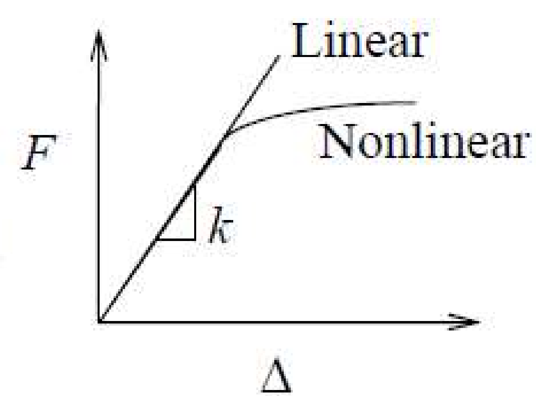
Nodal forces: f_i , f_j

Spring constant (stiffness): k

Spring force-displacement relationship:

$$F = k\Delta \qquad \text{with } \Delta = u_j - u_i$$

$k = F / \Delta$ is the force needed to produce a unit stretch.

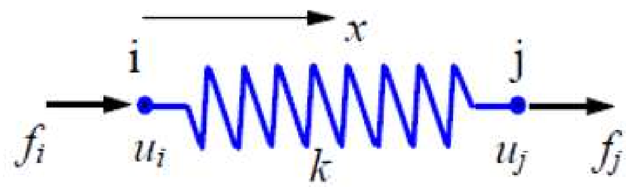


Consider the equilibrium of forces for the spring. At node i,

(Linear problems)

Consider the equilibrium of forces for the spring. At node i, we have

$$f_i = -F = -k(u_j - u_i) = ku_i - ku_j$$



and at node j,

$$f_j = F = k(u_j - u_i) = -ku_i + ku_j$$

In matrix form,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

or,

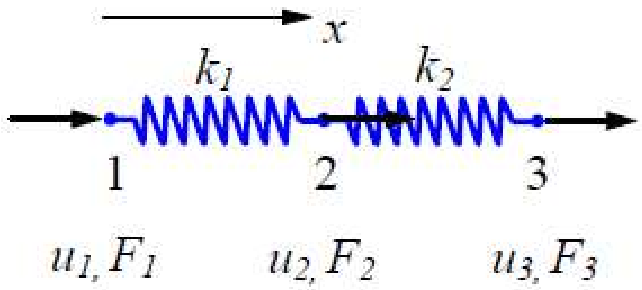
$$\mathbf{ku} = \mathbf{f}$$

$\mathbf{k}$ = (element) stiffness matrix

$\mathbf{u}$ = (element nodal) displacement vector

$\mathbf{f}$ = (element nodal) force vector

Springs connected in series



For element 1,

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix}$$

element 2,

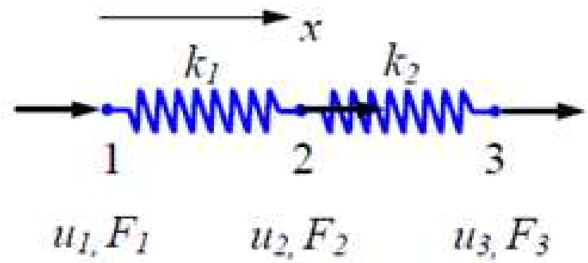
$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^2 \\ f_2^2 \end{Bmatrix}$$

where f_i^m is the (internal) force acting on *local* node i of element m ($i = 1, 2$).

Assemble the stiffness matrix for whole system

Consider the equilibrium of forces at node 1,

$$F_1 = f_1^1$$



at node 2,

$$F_2 = f_2^1 + f_1^2$$

and node 3,

$$F_3 = f_2^2$$

$$F_1 = k_1 u_1 - k_1 u_2$$
$$F_2 = -k_1 u_1 + (k_1 + k_2) u_2 - k_2 u_3$$
$$F_3 = -k_2 u_2 + k_2 u_3$$

$$F_1 = k_1 u_1 - k_1 u_2$$
$$F_2 = -k_1 u_1 + (k_1 + k_2) u_2 - k_2 u_3$$
$$F_3 = -k_2 u_2 + k_2 u_3$$

Matrix Form

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

or

$$\mathbf{KU} = \mathbf{F}$$

K is the stiffness matrix (structure matrix) for the spring system.

Alternative way of assembling whole stiffness matrix

For element 1,

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix}$$

element 2,

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^2 \\ f_2^2 \end{Bmatrix}$$

"Enlarging" the stiffness matrices for elements 1 and 2

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_1^2 \\ f_2^2 \end{Bmatrix}$$

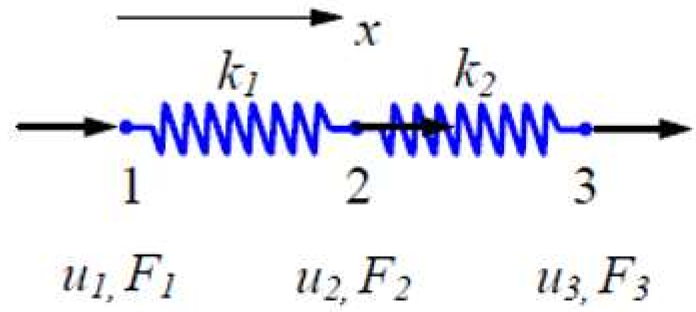
This is the same equation we derived by using the force equilibrium concept.

Adding the two matrix equations (*superposition*),

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 \end{Bmatrix}$$

This is the same, derived using the force equilibrium concept.

Boundary conditions and load conditions



Assuming, $u_1 = 0$ and $F_2 = F_3 = P$

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ P \end{Bmatrix}$$

which reduces to

Deleting the 1st row and column

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ P \end{Bmatrix}$$

and

$$F_1 = -k_1 u_2$$

Unknowns are

$$\mathbf{U} = \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$ and the reaction force F_1 (if desired).

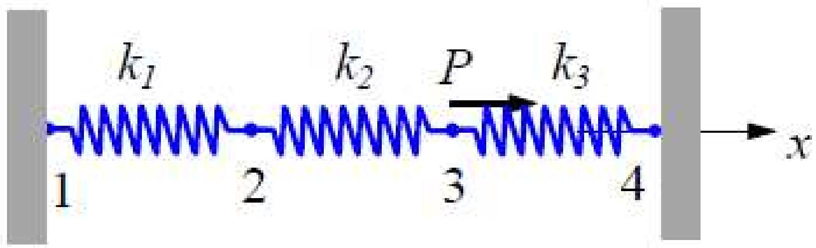
- Solving equations, displacements obtained

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2P/k_1 \\ 2P/k_1 + P/k_2 \end{Bmatrix}$$

and the reaction force

$$F_1 = -2P$$

Numerical on springs connected in series



Given: For the spring system shown above,

$$k_1 = 100 \text{ N / mm}, \;\; k_2 = 200 \text{ N / mm}, \;\; k_3 = 100 \text{ N / mm}$$

$$P = 500 \text{ N}, \;\; \text{u}_1 = u_4 = 0$$

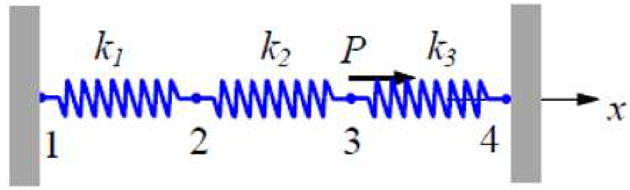
Find: (a) the global stiffness matrix

(b) displacements of nodes 2 and 3

(c) the reaction forces at nodes 1 and 4

(d) the force in the spring 2

Solution



(a) The element stiffness matrices are

$$\mathbf{k}_1 = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \quad (\text{N/mm}) \qquad (1)$$

For element 1,

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$\mathbf{k}_2 = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \quad (\text{N/mm})$$

$$\mathbf{k}_3 = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \quad (\text{N/mm}) \qquad (3)$$

Applying the superposition concept, global stiffness matrix obtained as

$$\mathbf{K} = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+200 & -200 & 0 \\ 0 & -200 & 200+100 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \end{array}$$

$$\mathbf{K} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix}$$

which is *symmetric* and ***banded***.

Equilibrium (FE) equation for the whole system is

$$\begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ P \\ F_4 \end{Bmatrix} \qquad (4)$$

(b) Applying the BC($u_1 = u_4 = 0$) in Eq(4), or deleting 1st and 4th rows and column

$$\begin{bmatrix} 300 & -200 \\ -200 & 300 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \qquad (5)$$

Solving Eq(5)

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P/250 \\ 3P/500 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \text{ (mm)} \qquad (6)$$

(c) From the 1st and 4th equations in (4), we get the reaction forces

$$F_1 = -100u_2 = -200 \quad \text{(N)}$$

$$F_4 = -100u_3 = -300 \quad \text{(N)}$$

(d) The FE equation for spring (element) 2 is

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

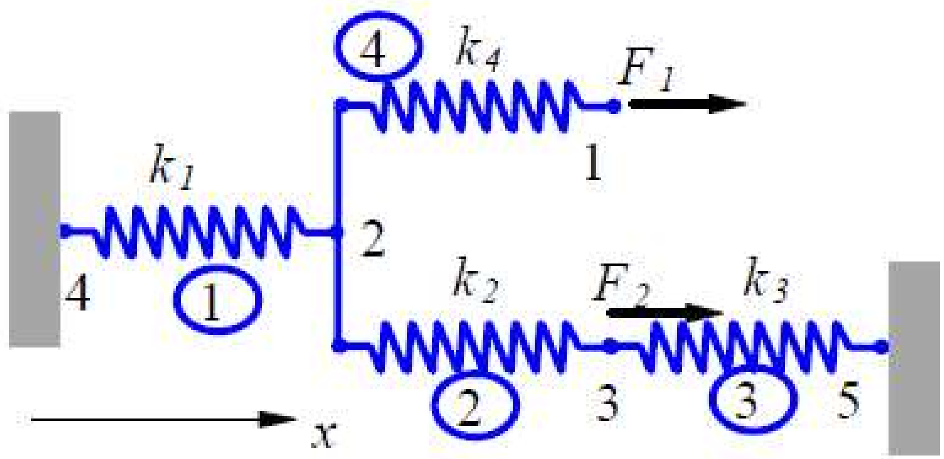
Here $i = 2$, $j = 3$ for element 2. Thus we can calculate the spring force as

$$F = f_j = -f_i = \begin{bmatrix} -200 & 200 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} -200 & 200 \end{bmatrix} \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

$$= 200 \quad \text{(N)}$$

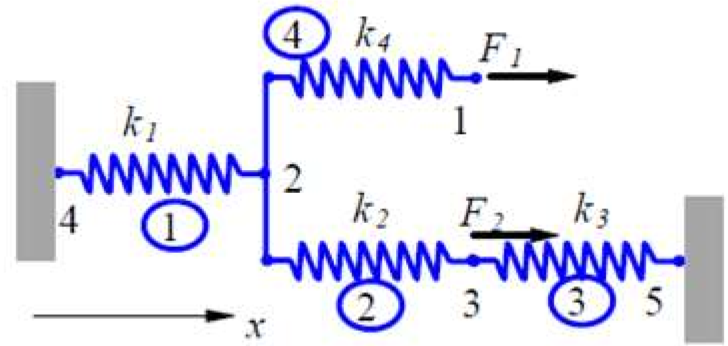
Springs connected in series and parallel



Problem: For the spring system with arbitrarily numbered nodes and elements, as shown above, find the global stiffness matrix.

***Solution*:**

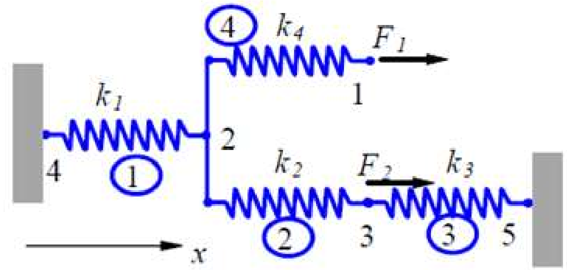
First we construct the following



Element Connectivity Table

Element	*Node i (1)*	*Node j (2)*
1	4	2
2	2	3
3	3	5
4	2	1

which specifies the *global* node numbers corresponding to the *local* node numbers for each element.



Then we can write the element stiffness matrices as follows

$$\mathbf{k}_1 = \begin{array}{cc} u_4 & u_2 \\ \end{array} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \qquad \mathbf{k}_2 = \begin{array}{cc} u_2 & u_3 \\ \end{array} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\mathbf{k}_3 = \begin{array}{cc} u_3 & u_5 \\ \end{array} \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \qquad \mathbf{k}_4 = \begin{array}{cc} u_2 & u_1 \\ \end{array} \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}$$

Finally, applying the superposition method, we obtain the global stiffness matrix as follows

$$\mathbf{K} = \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} k_4 & -k_4 & 0 & 0 & 0 \\ -k_4 & k_1 + k_2 + k_4 & -k_2 & -k_1 & 0 \\ 0 & -k_2 & k_2 + k_3 & 0 & -k_3 \\ 0 & -k_1 & 0 & k_1 & 0 \\ 0 & 0 & -k_3 & 0 & k_3 \end{bmatrix} \end{array}$$

The matrix is *symmetric*, *banded*, but *singular*.

References

- **The Finite element Method for Engineers – K. H. Hucbner, John Wiley and Sons**
- **Finite element Method- Y. M. Desai, T. I. Eldho & A. H. Shah, Pearson**
- **Introduction to Finite element Method- Yijun Lui**
- **An Introduction of Finite Element Method- J. N. Reddy , McGraw – Hill**
- **Applied Finite element analysis for engineers- Frank L. Stasa, CBS International**


THANK YOU

Direct Approach–Bar Element

Lecture09

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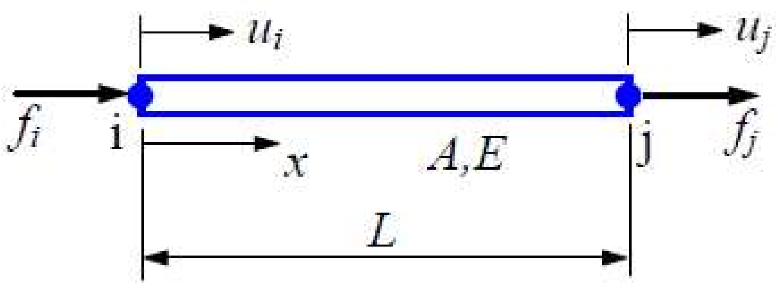
Bar Element

Consider a uniform prismatic bar:



L	length
A	cross-sectional area
E	elastic modulus
$u = u(x)$	displacement
$\varepsilon = \varepsilon(x)$	strain
$\sigma = \sigma(x)$	stress

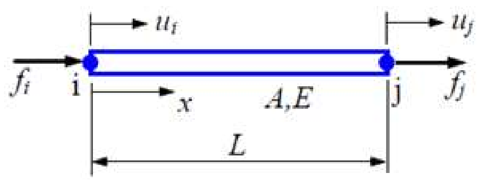
Strain-displacement relation:

$$\varepsilon = \frac{du}{dx} \qquad \ldots(1)$$

Stress-strain relation:

$$\sigma = E\varepsilon \qquad \ldots(2)$$

Stiffness Matrix



Assuming that the displacement u is *varying linearly* alon the axis of the bar, i.e.,

$$u(x) = \left(1 - \frac{x}{L}\right) u_i + \frac{x}{L} u_j \qquad (3)$$

we have

$$\varepsilon = \frac{u_j - u_i}{L} = \frac{\Delta}{L} \qquad (\Delta = \text{elongation}) \qquad (4)$$

$$\sigma = E\varepsilon = \frac{E\Delta}{L} \qquad (5)$$

$$\sigma = \frac{F}{A} \qquad (F = \text{force in bar}) \qquad (6)$$

Thus, (5) and (6) lead to

$$F = \frac{EA}{L}\Delta = k\Delta \qquad (7)$$

where $k = \dfrac{EA}{L}$ is the stiffness of the bar.

The bar is acting like a spring in this case and we conclude that element stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}$$

or

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (8)$$

This can be verified by considering the equilibrium of the forces at the two nodes.

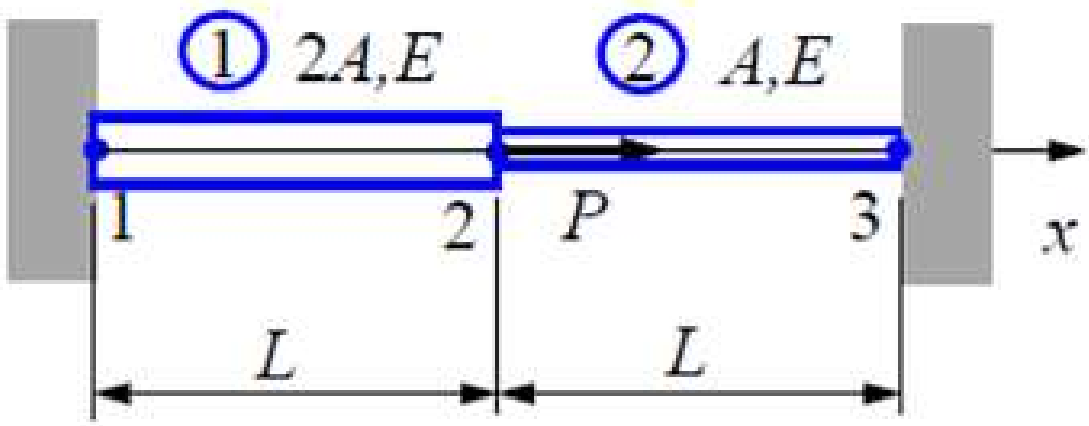
Element equilibrium equation is

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} \qquad (9)$$

$$\mathbf{ku} = \mathbf{f}$$

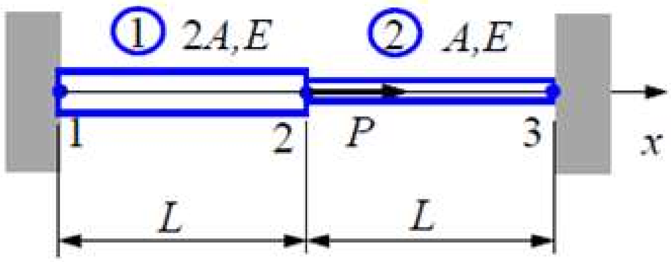
Numerical Problem

Find the stresses in the two bar assembly which is loaded with force P, and constrained at the two ends, as shown in the figure.



Solution

- Use two 1-D bar element



Element 1,

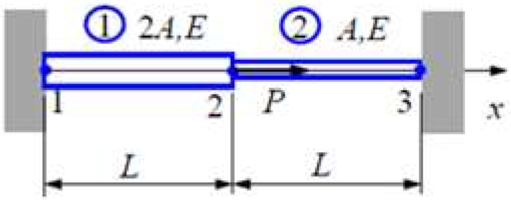
$$\mathbf{k}_1 = \frac{2EA}{L}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Element 2,

$$\mathbf{k}_2 = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Imagine a frictionless pin at node 2, which connects the two elements. We can assemble the global FE equation as follows,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$



Load and boundary conditions (BC) are,

$$u_1 = u_3 = 0, \qquad F_2 = P$$

FE equation becomes,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

Deleting the 1st row and column, and the 3rd row and column, we obtain,

$$\frac{EA}{L}[3]\{u_2\} = \{P\}$$

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

Thus,

$$u_2 = \frac{PL}{3EA}$$

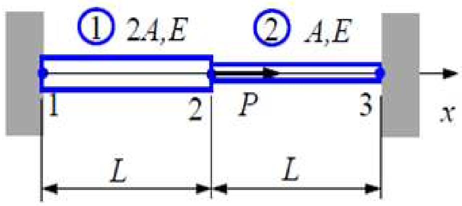
and

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{3EA}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Stress in element 1 is

$$\sigma_1 = E\varepsilon_1 = E\mathbf{B}_1\mathbf{u}_1 = E\begin{bmatrix}-1/L & 1/L\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$= E\frac{u_2 - u_1}{L} = \frac{E}{L}\left(\frac{PL}{3EA} - 0\right) = \frac{P}{3A}$$



Similarly, stress in element 2 is

$$\sigma_2 = E\varepsilon_2 = E\mathbf{B}_2\mathbf{u}_2 = E\begin{bmatrix}-1/L & 1/L\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}$$

$$= E\frac{u_3 - u_2}{L} = \frac{E}{L}\left(0 - \frac{PL}{3EA}\right) = -\frac{P}{3A}$$

which indicates that bar 2 is in compression.

ONE-DIMENSIONAL TORSION OF A CIRCULAR SHAFT

- The one-dimensional torsional elements can be used to analyze an assemblage of circular shafts.
  - A two-node torsion element is shown in Fig. The $T1$ and $T2$ are the torques at nodes 1 and 2, respectively, whereas ø1 and ø2 are angular rotations at nodes 1 and 2, respectively, of the element.

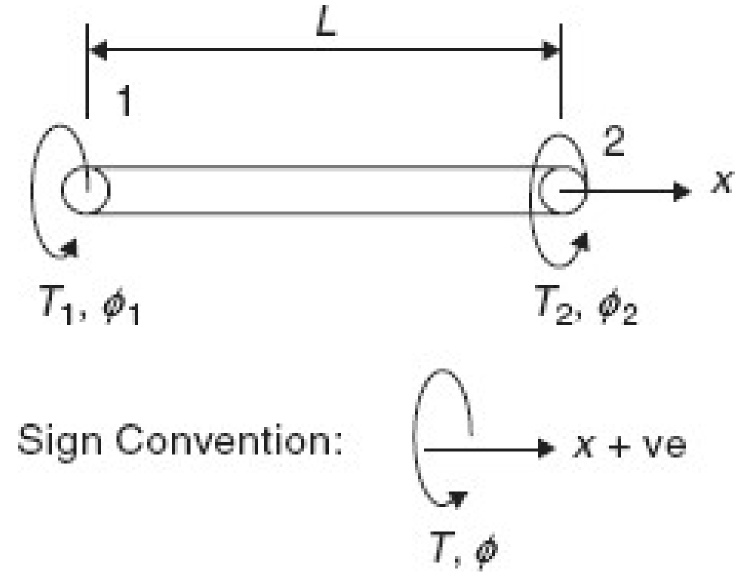


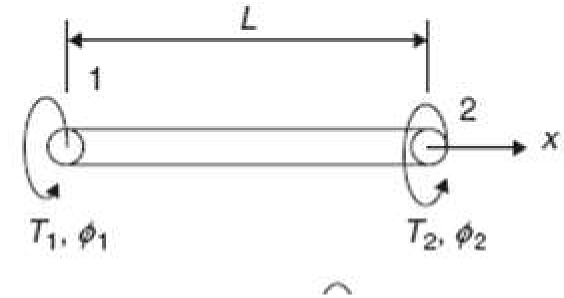
Fig. 1-D Torsion element

Constitutive Law

$$T = GJ\frac{d\phi}{dx}$$

where G = shear modulus and J = polar moment of inertia of the cross-sectional area about the axis of the element.

$$T = GJ\frac{d\phi}{dx} \Rightarrow T = GJ\frac{\Delta\phi}{L}$$



For equilibrium, $T_2 = -T_1 = T$

at node 1

$$T_1 = -T = \frac{GJ}{L}(\phi_1 - \phi_2)$$

At node 2

$$T_2 = T = \frac{GJ}{L}(\phi_2 - \phi_1)$$

$$T_2 = \frac{GJ}{L}(-\phi_1 + \phi_2)$$

k^e = (elemental) stiffness matrix

$\boldsymbol{\emptyset}$ = (elemental nodal) angular rotation vector

$\boldsymbol{T}$ = (elemental nodal) torque vector

In Matrix form

$$\frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \Rightarrow k^e\phi = T$$

$$k^e = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Numerical Problem Torsion of a circular shaft

- A circular shaft as shown in Figure is subjected to torques $T2$ and $T3$ as shown in the diagram. By employing one-dimensional torsion elements, compute angular rotations at nodes 2 and 3 and reactive torques at nodes 1 and 4.

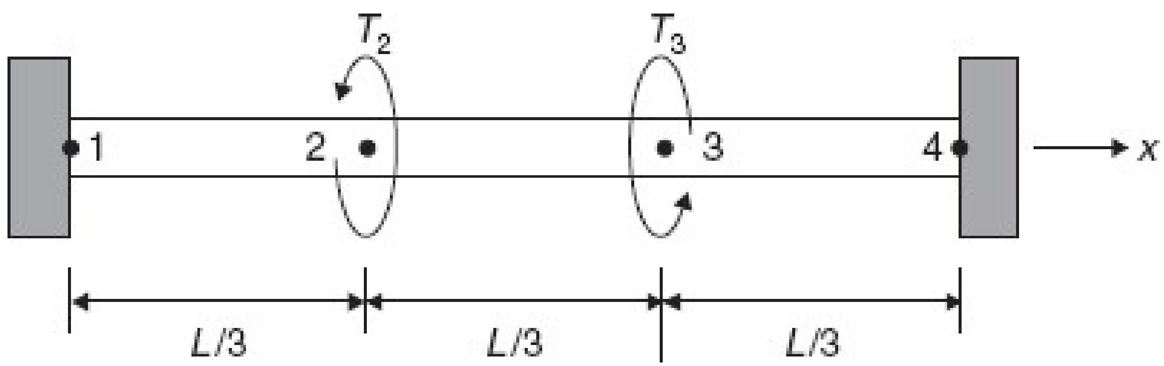
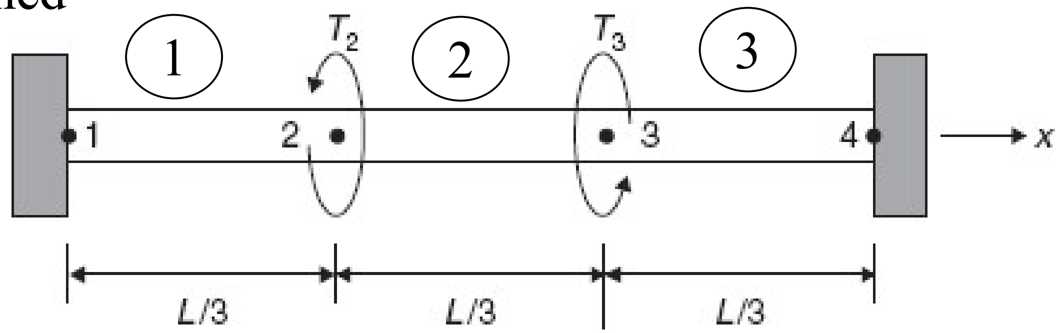


Fig. Circular shaft

1

Solution

- The element matrices can be obtained as follows.



$$\underline{k}^{(1)} = \frac{3GJ}{L} \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

$$\underline{k}^{(2)} = \frac{3GJ}{L} \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

$$\underline{k}^{(3)} = \frac{3GJ}{L} \begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

The system can be assembled as

$$\frac{3GJ}{L} \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array} \begin{Bmatrix} \phi_1 = 0 \\ \phi_2 = ? \\ \phi_3 = ? \\ \phi_4 = 0 \end{Bmatrix} = \begin{Bmatrix} T_1 = ? \\ -T_2 \\ T_3 \\ T_4 = ? \end{Bmatrix}$$

- By applying homogeneous boundary conditions ø1 = ø4 = 0, following equation is obtained.

$$\frac{3GJ}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -T_2 \\ T_3 \end{Bmatrix}$$

- After solving equations, the solution is obtained as,

$$\begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \frac{L}{9GJ}\begin{Bmatrix} T_3 - 2T_2 \\ 2T_3 - T_2 \end{Bmatrix}$$

Reactive torques T1 and T4?

$$\phi_2 = \frac{L}{9GJ}(T_3 - 2T_2)$$

$$\phi_3 = \frac{L}{9GJ}(2T_3 - T_2)$$

$$\frac{3GJ}{L} \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array} \begin{Bmatrix} \phi_1 = 0 \\ \phi_2 = ? \\ \phi_3 = ? \\ \phi_4 = 0 \end{Bmatrix} = \begin{Bmatrix} T_1 = ? \\ -T_2 \\ T_3 \\ T_4 = ? \end{Bmatrix}$$

$$\frac{3GJ}{L}(-\phi_2) = T_1$$

Putting the value of ϕ_2

$$T_1 = \frac{1}{3}(T_3 - 2T_2)$$

$$\frac{3GJ}{L}(-\phi_3) = T_4$$

Putting the value of ϕ_3

$$T_4 = \frac{1}{3}(T_2 - 2T_3)$$

References

- **The Finite element Method for Engineers – K. H. Hucbner, John Wiley and Sons**
- **Finite element Method- Y. M. Desai, T. I. Eldho & A. H. Shah, Pearson**
- **Introduction to Finite element Method- Yijun Lui**
- **An Introduction of Finite Element Method- J. N. Reddy , McGraw – Hill**
- **Applied Finite element analysis for engineers- Frank L. Stasa, CBS International**


THANK YOU

Direct Approach- Heat transfer….
Lecture10

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One-dimensional Steady State Heat Conduction

- Taking a simple case of a section of layered material through heat is flowing only in x-direction.
- In this heat conduction problem, assuming that there is no internal heat generation, that the left hand side of the element is held at a uniform temperature higher than that of right hand side.
- Each element is homogeneous solid whose thermal conductivity is known in the direction of heat flow.
- Heat flux temperature, thermal conductivity and layer thickness are concerned parameter.
- This problem spilts into a series of simpler ones by considering each layer of material as finite element whose characteristics can be determined by the basic law of heat conduction.
- The field variable for this problem is the temperature.

- A 2-node heat flow element as shown in Figure can be utilized to analyze one-dimensional steady state heat conduction problems. Note that convection is neglected in the formulation.

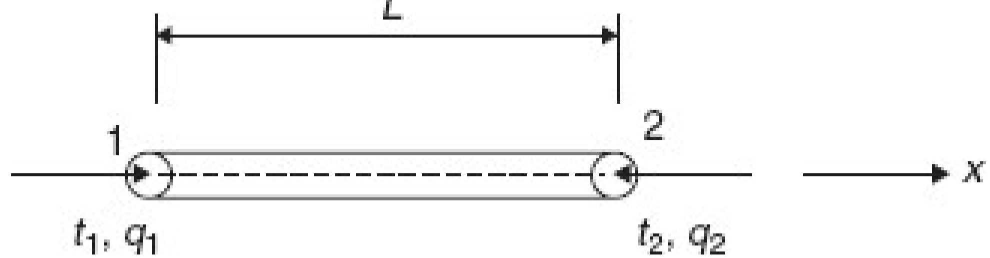


Figure -One-dimensional heat flow Element

- Heat flow entering a control volume is positive. Outgoing heat flow is negative.
- Let q be the heat flow (per unit area) and t be the temperature. Then, q_i, $i = 1, 2$ are the nodal heat flow rates and t_i, $i = 1, 2$ are the nodal temperatures

- Gradient of Unknown Temperature

$$\frac{dt}{dx} = \frac{t_2 - t_1}{L}$$

- Constitutive Law: Fourier's Law of Heat Conduction
    - According to Fourier's law, the heat flow in the *x* direction is given by

$$q = -K_{xx}\frac{dt}{dx}$$

$$= -K_{xx}\frac{(t_2 - t_1)}{L} = k(t_1 - t_2) \qquad \text{....(1)}$$

where $k = \frac{K_{xx}}{L}$ and K_{xx}= Thermal conductivity in *x*-direction [KW/m.° C in SI units].

- The K_{xx} measures the amount of heat energy (W.hr) that flows through a unit length of a given substance in a unit time (hr.) to raise the temperature 1 degree (° C).

▪ Element equations

- Since physics of the element can be understood easily for the one-dimensional heat flow element, direct equilibrium approach can be employed to derive element equations. By employing Eq. (1), the heat flowing in the *x*-direction at node 1 is

$$q_1 = k(t_1 - t_2)$$

- From the principle of conservation of energy, the net heat flow into the system must be zero for a steady state condition.

$$\Rightarrow \quad q_1 + q_2 = 0 \quad \text{or} \quad q_2 = -q_1$$

- The element equations are given by

$$\begin{aligned} q_1 &= k(t_1 - t_2) \\ q_2 &= k(-t_1 + t_2) \end{aligned} \quad \Rightarrow \quad k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Numerical Problem on 1-D heat transfer

- The furnace wall shown in the Figure 1 consists of 25 cm of fire brick [$K1$ = 0.012 W/(cm ° C)], 10 cm of insulation brick [$K2$ = 0.0014 W/(cm ° C)], and 20 cm of red brick [$K3$ = 0.0086 W/(cm ° C)]. The specified inner and outer temperatures are 500° C and 150° C, respectively. Determine the internal temperature distribution.

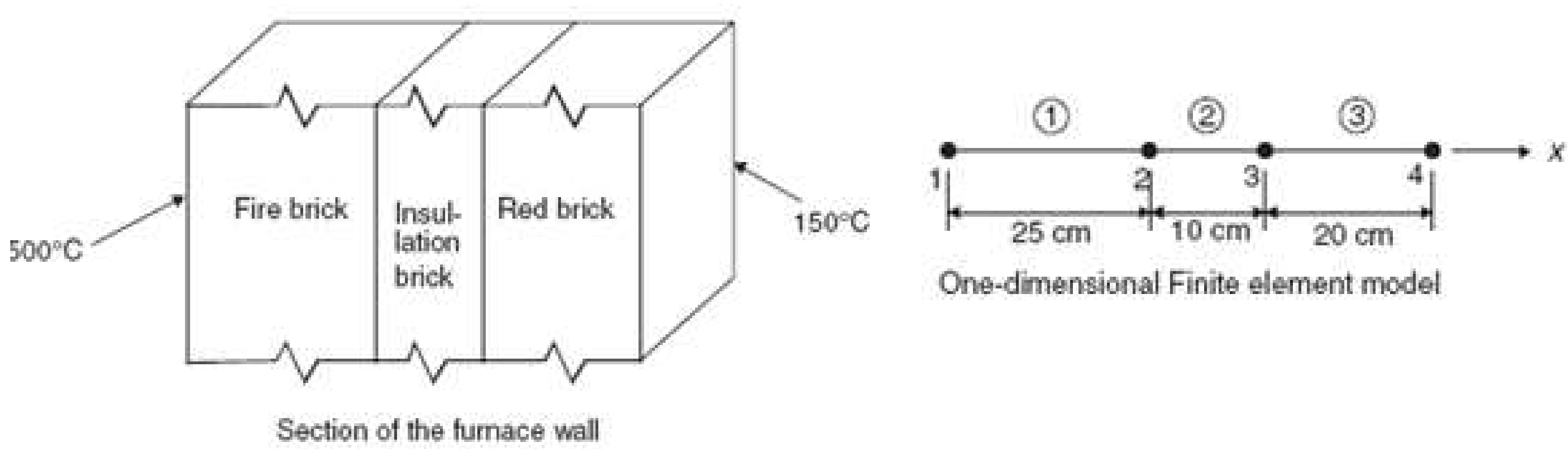


Figure 1 Cross section of furnace wall and its finite element model

Solution:

- This problem can be treated as a one-dimensional model. A constant arbitrary heat conduction area of unit cm2 is assumed for all elements.
- Element conduction matrices are given by

$$\underline{k}^{(1)} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array} \frac{0.012}{(25)}$$

$$\underline{k}^{(2)} = \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array} \frac{0.0014}{(10)}$$

$$\underline{k}^{(3)} = \begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array} \frac{0.0086}{(20)}$$

Assembly

$$10^{-4}\begin{bmatrix} 4.8 & -4.8 & 0 & 0 \\ -4.8 & (4.8+1.4) & -1.4 & 0 \\ 0 & -1.4 & 1.4+4.3 & -4.3 \\ 0 & 0 & -4.3 & 4.3 \end{bmatrix}\begin{Bmatrix} t_1 = 500 \\ t_2 = ? \\ t_3 = ? \\ t_4 = 150 \end{Bmatrix} = \begin{Bmatrix} q_1 = ? \\ 0 \\ 0 \\ q_4 = ? \end{Bmatrix}$$

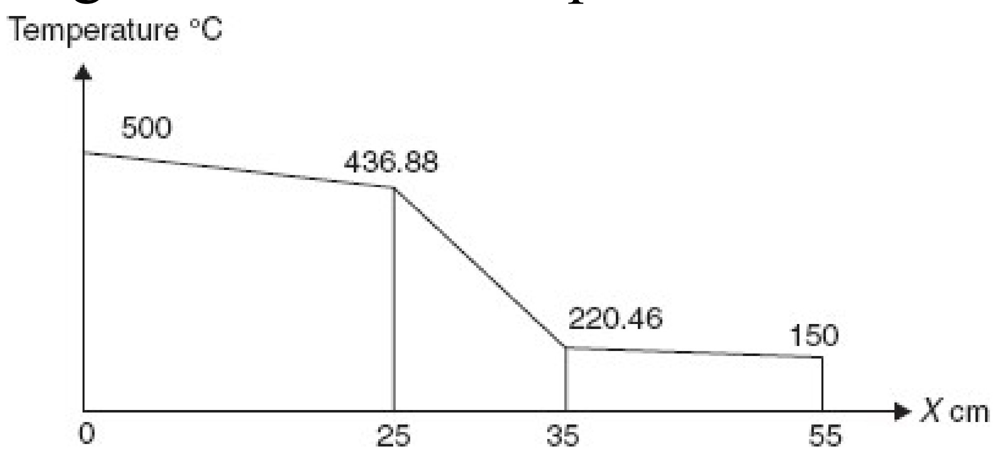
- By applying the non-homogeneous boundary conditions, the following equation can be obtained.

$$10^{-4}\begin{bmatrix} 6.2 & -1.4 \\ -1.4 & 5.7 \end{bmatrix}\begin{Bmatrix} t_2 \\ t_3 \end{Bmatrix} = (-10^{-4})\begin{bmatrix} -4.8 & 0 \\ 0 & -4.3 \end{bmatrix}\begin{Bmatrix} 500 \\ 150 \end{Bmatrix} = \begin{Bmatrix} 2,400 \\ 645 \end{Bmatrix}10^{-4}$$

Hence,

$$\begin{Bmatrix} t_2 \\ t_3 \end{Bmatrix} = \frac{1}{33.38}\begin{bmatrix} 5.7 & 1.4 \\ 1.4 & 6.2 \end{bmatrix}\begin{Bmatrix} 2,400 \\ 645 \end{Bmatrix} = \begin{Bmatrix} 436.88 \\ 220.46 \end{Bmatrix}{}^{\circ}C$$

- Figure shows the temperature distribution.



$$q_1 = -10^{-4}[4.8\times 500 - 4.8\times 436.88] = 0.0303 \ \text{w/cm}^2$$

$$q_4 = -10^{-4}[-4.3\times 220.46 + 4.3\times 150] = -0.0303 \ \text{w/cm}^2$$

1-D Fluid FLOW

Analysis for one-dimensional flow through porous media is similar to the heat flow analysis (e.g., ground water flow, seepage). A 2-node element as shown in Figure is the simplest possible element can be used to analyze one-dimensional seepage problems.

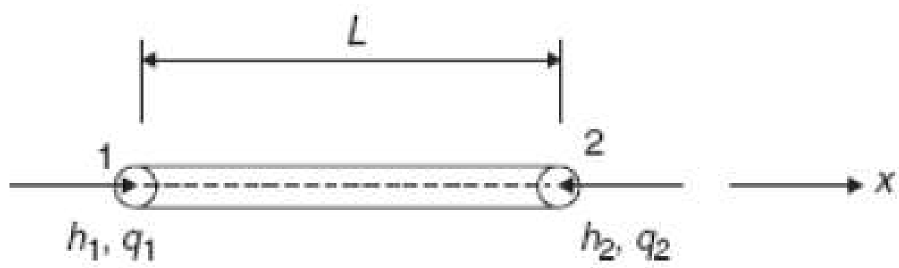


Figure- 1-D Fluid flow element

Gradient of hydraulic head

- The gradient is obtained as:

$$\frac{dh}{dx} = \frac{h_2 - h_1}{L}$$

Constitutive Law: (Darcy's law)

- According to the Darcy's law; the fluid velocity $\boldsymbol{v_x}$ in the x direction is given by

$$v_x = -K_{xx}\frac{dh}{dx}$$

$$= -K_{xx}\frac{(h_2 - h_1)}{L} = K_{xx}\frac{(h_1 - h_2)}{L} \quad \ldots.(1)$$

where $\boldsymbol{K_{xx}}$ is the coefficient of permeability in the x-direction.

- The volumetric flow rate q can be obtained from

$$q = Av_x = AK_{xx}\frac{(h_1 - h_2)}{L} \quad \ldots.(2)$$

A represents cross-sectional area of the element.

Element Equations

- The element equations can be derived easily by adopting the direct equilibrium approach. From Eq. (2) fluid flow entering the element in the x-direction at node 1 is

$$q_1 = k(h_1 - h_2)$$

where

$$k = \frac{AK_{xx}}{L}$$

- From the principle of conservation of mass, the net fluid flow into the system must be zero for a steady state condition.

$$q_1 + q_2 = 0 \qquad \text{or} \qquad q_2 = -q_1$$

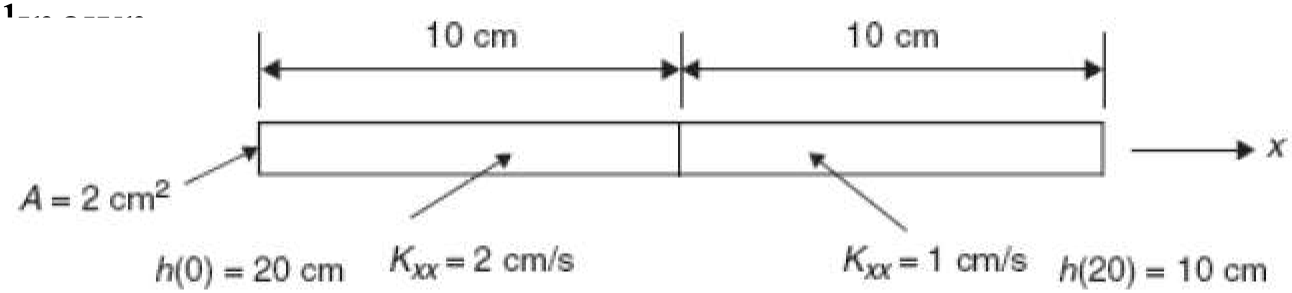
- The element equations are given by

$$q_1 = k(h_1 - h_2)$$
$$q_2 = k(-h_1 + h_2)$$

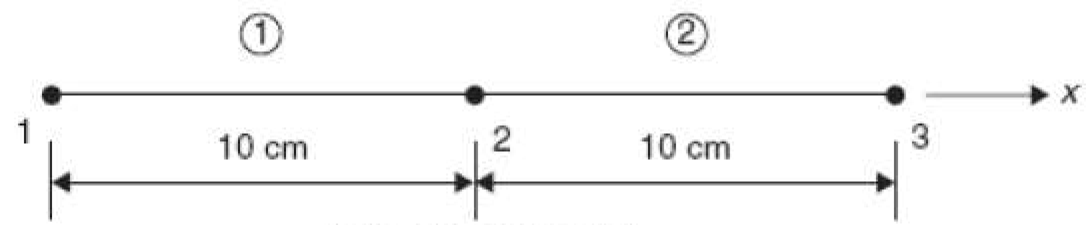
$$\Rightarrow \quad k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Numerical Problem on fluid flow

- In a laboratory, experiment of one-dimensional flow through porous media over the section shown, the following data are



One-dimensional Flow through porous media



Finite element model

- Determine the distribution of hydraulic head over the length of the section, the flow rate and fluid velocities in each element.

Solution

Element matrices are given by

$$\underline{k}^{(1)} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array} \frac{2\times 2}{10}$$

$$\underline{k}^{(2)} = \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array} \frac{1\times 2}{10}$$

Assembly

$$\frac{2}{10}\begin{bmatrix} 2 & -2 & 0 \\ -2 & (2+1) & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} h_1 = 20 \\ h_2 = ? \\ h_3 = 10 \end{Bmatrix} = \begin{Bmatrix} q_1 = ? \\ 0 \\ q_3 = ? \end{Bmatrix}$$

Hence,

$$0.6h_2 = 0.4\times 20 + 0.2\times 10$$

or

$$h_2 = 16.6667 \text{ cm}$$

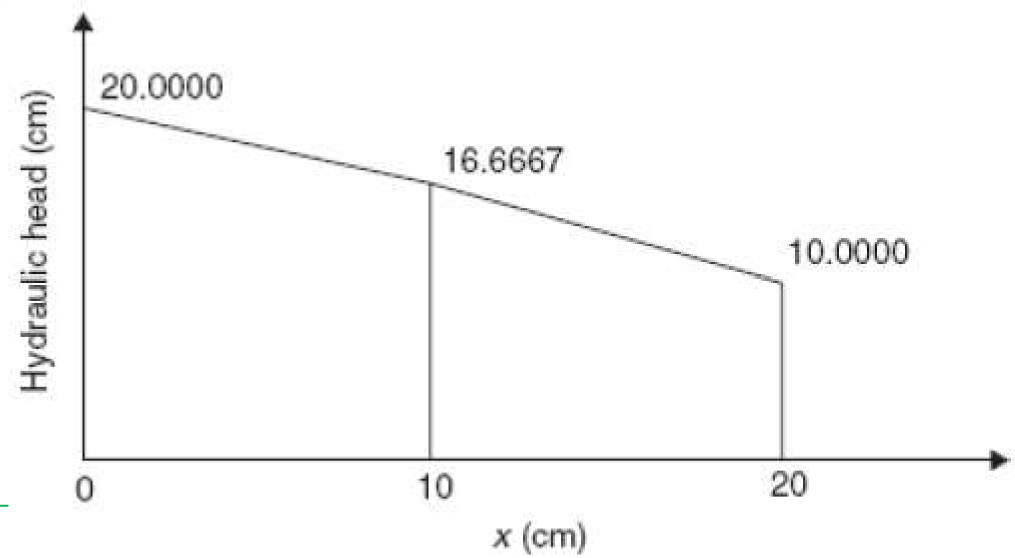
$$q_1 = 2(0.2\times 20 - 0.2\times 16.67) = 1.333\ \text{cm}^3/\text{s}$$
$$q_3 = 2(-0.1\times 16.67 + 0.1\times 10) = -1.333\ \text{cm}^3/\text{s}$$

Flow rate = 1.333 cm^3/s.

- By making use of Eq. (1), fluid velocity in each element can be obtained as

$$v_x^{(1)} = \frac{2}{10}(h_1 - h_2)$$
$$= 0.2\times(20 - 16.6667) = 0.6667\ \text{cm/s}$$
$$v_x^{(2)} = \frac{1}{10}(h_2 - h_3)$$
$$= 0.1\times(16.6667 - 10) = 0.6667\ \text{cm/s}$$

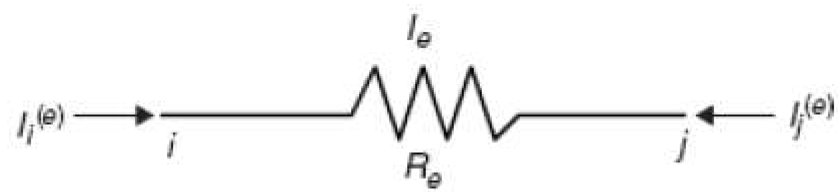
The hydraulic head distribution is shown in Figure



Electrical Network Analysis

- In the previous section(1-D fluid flow), the fluid flow network was discussed, wherein flow path was considered as a finite element of fluid network. Similarly, the direct-current electrical networks can be considered as a finite element network.
- The current carrying member of the electrical network can be taken as a finite element. The Ohm's law can be used for establishing the element characteristics.
- Procedure of finite element formulation is very similar to those used for fluid flow network. Voltages $V1$ and $V2$ play the same role as the nodal pressures and a current I replaces the flow rate Q.
- Basic relationship between the current flowing and voltage difference as given by Ohms' law is

$$I_e = \frac{\Delta V}{R_e} = G_e(V_i - V_j)$$



Summary

References

- **The Finite element Method for Engineers – K. H. Huebner, John Wiley and Sons**
- **Finite element Method- Y. M. Desai, T. I. Eldho & A. H. Shah, Pearson**
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