Control Systems

Subject Code: BEC-26

Third Year ECE

Unit-II

Shadab A. Siddique
Assistant Professor

Maj. G. S. Tripathi
Associate Professor

Lecture 6

Department of Electronics & Communication Engineering,
Madan Mohan Malaviya University of Technology, Gorakhpur

THE DESIGN OF STATE VARIABLE FEEDBACK SYSTEMS

The time-domain method, expressed in terms of state variables, can also be utilized to design a suitable compensation scheme for a control system. Typically, we are interested in controlling the system with a control signal, u(t), which is a function of several measurable state variables. Then we develop a state variable controller that operates on the information available in measured form.

State variable design is typically comprised of three steps. In the first step, we assume that all the state variables are measurable and utilize them in a **full-state feedback control law**. Full-state feedback is not usually practical because it is not possible (in general) to measure all the states. In paractice, only certain states (or linear combinations thereof) are measured and provided as system outputs. The second step in state varaible design is to construct an **observer** to estimate the states that are not directly sensed and available as outputs. Observers can either be full-state observers or reduced-order observers. Reduced-order observers account for the fact that certain states are already available as system outputs; hence they do not need to be estimated. The final step in the design process is to appropriately connect the observer to the full-state feedback conrol low. It is common to refer to the state-varaible controller as a compensator. Additionally, it is possible to consider reference inputs to the state variable compensator to complete the design.

CONTROLLABILITY:

Full-state feedback design commonly relies on **pole-placement techniques**. It is important to note that a system must be completely controllable and completely observable to allow the flexibility to place all the closed-loop system poles arbitrarily. The concepts of controllability and observability were introduced by Kalman in the 1960s.

A system is completely controllable if there exists an unconstrained control u(t) that can transfer any initial state $x(t_0)$ to any other desired location x(t) in a finite time, $t_0 \leq t \leq T$.

A control system is said to be **controllable** if the initial states of the control system are transferred (changed) to some other desired states by a controlled input in finite duration of time.

We can check the controllability of a control system by using **Kalman's test**.

- Write the matrix Q_c in the following form.

$$Q_c = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$$

- Find the determinant of matrix Q_c and if it is not equal to zero, then the control system is controllable.

For the system

$$\dot{x} = Ax + Bu$$

we can determine whether the system is controllable by examining the algebraic condition

$$\text{rank}\begin{bmatrix} B & AB & A^2B \cdots A^{n-1}B \end{bmatrix} = n$$

The matrix A is an nxn matrix an B is an nx1 matrix. For multi input systems, B can be nxm, where m is the number of inputs.

For a single-input, single-output system, the controllability matrix P_c is described in terms of A and B as

$$P_c = \begin{bmatrix} B & AB & A^2B \cdots A^{n-1}B \end{bmatrix}$$

which is nxn matrix. Therefore, if the determinant of P_c is nonzero, the system is controllable.

Example:

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, AB = \begin{bmatrix} 0 \\ 1 \\ -a_2 \end{bmatrix}, A^2B = \begin{bmatrix} 1 \\ -a_2 \\ (a_2^2 - a_1) \end{bmatrix}$$

$$P_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & (a_2^2 - a_1) \end{bmatrix}$$

The determinant of P_c =1 and ≠0 , hence this system is controllable.

Example.

Consider a system represented by the two state equations

$$\dot{x}_1 = -2x_1 + u \, , \, \dot{x}_2 = -3x_2 + d\,x_1$$

The output of the system is $y=x_2$. Determine the condition of controllability.

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } AB = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ d \end{bmatrix}$$

$$P_c = \begin{bmatrix} 1 & -2 \\ 0 & d \end{bmatrix}$$

The determinant of pc is equal to d, which is nonzero only when d is nonzero.

OBSERVABILITY:

All the poles of the closed-loop system can be placed arbitrarily in the complex plane if and only if the system is **observable** and **controllable**. Observability refers to the ability to estimate a state variable.

A system is completely observable if and only if there exists a finite time T such that the initial state x(0) can be determined from the observation history y(t) given the control u(t).

Consider the single-input, single-output system

$$\dot{x} = Ax + Bu \quad \text{and} \quad y = Cx$$

where C is a 1xn row vector, and x is an nx1 column vector. This system is completely observable when the determinant of the **observability matrix** $\mathbf{P_0}$ is nonzero.

The observability matrix, which is an nxn matrix, is written as

$$P_O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

A control system is said to be **observable** if it is able to determine the initial states of the control system by observing the outputs in finite duration of time.

We can check the observability of a control system by using **Kalman's test**.

- Write the matrix Q_o in following form.

$$Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \ldots \quad (A^T)^{n-1} C^T]$$

- Find the determinant of matrix Q_o and if it is not equal to zero, then the control system is observable.

Example:

Consider the previously given system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad , \quad CA^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Thus, we obtain

$$P_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The det P_0=1, and the system is completely observable. Note that determination of observability does not utility the B and C matrices.

Example: Consider the system given by

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

We can check the system controllability and observability using the P_c and P_0 matrices.

From the system definition, we obtain

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Therefore, the controllability matrix is determined to be

$$P_c = [B \quad AB] = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

det P_c=0 and rank(P_c)=1. Thus, the system is not controllable.

From the system definition, we obtain

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad CA = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Therefore, the observability matrix is determined to be

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

det P_O=0 and rank(P_O)=1. Thus, the system is not observable.

If we look again at the state model, we note that

$$y = x_1 + x_2$$

However,

$$\dot{x}_1 + \dot{x}_2 = 2x_1 + (x_2 - x_1) + u - u = x_1 + x_2$$

Example

Let us verify the controllability and observability of a control system which is represented in the state space model as,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \end{bmatrix}, D = [0] \quad and \quad n = 2$$

For $n = 2$, the matrix Q_c will be

$$Q_c = [B \quad AB]$$

We will get the product of matrices A and B as,

$$AB = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$|Q_c| = 1 \neq 0$$

Since the determinant of matrix Q_c is not equal to zero, the given control system is controllable.

For $n = 2$, the matrix Q_o will be -

$$Q_o = [C^T \quad A^T C^T]$$

Here,

$$A^T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad and \quad C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We will get the product of matrices A^T and C^T as

$$A^T C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Q_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow |Q_o| = -1 \quad \neq 0$$

Since, the determinant of matrix Q_o is not equal to zero, the given control system is observable.

Therefore, the given control system is both controllable and observable.

MODEL QUESTIONS

1. The System matrix of a continuous time system, described in the state variable form is

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

Determine the range of x & y so that the system is stable.

2. For a single input system

$$\dot{X} = AX + BU$$
$$Y = CX$$
$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ C = [1 \quad 1]$$

Check the controllability & observability of the system.

3. Given the homogeneous state space equation $\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} X$;

Determine the steady state value $X_{ss} = \lim_{t\to\infty} X(t)$ given the initial state value $X(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$.

4. State Kalman's test for observability.

The figures in the right-hand margin indicate marks.

5. For a system represented by the state equation $\dot{X} = AX$

Where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

Find the initial condition state vector X(0) which will excite only the mode corresponding to the eigenvalue with the most negative real part. **[10]**

6. Write short notes on Properties of state transition matrix. **[3.5]**

7. Investigate the controllability and observability of the following system:

$$\dot{X} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; Y = [0 \quad 1] X$$ **[8]**

UNIT-II

The End

Thank You

Shadab. A. Siddique

Maj. G. S. Tripathi