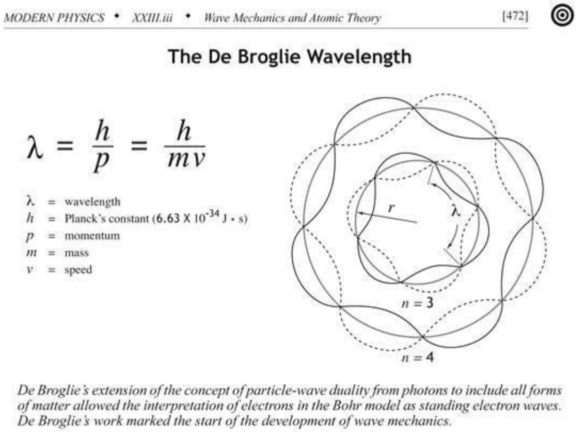
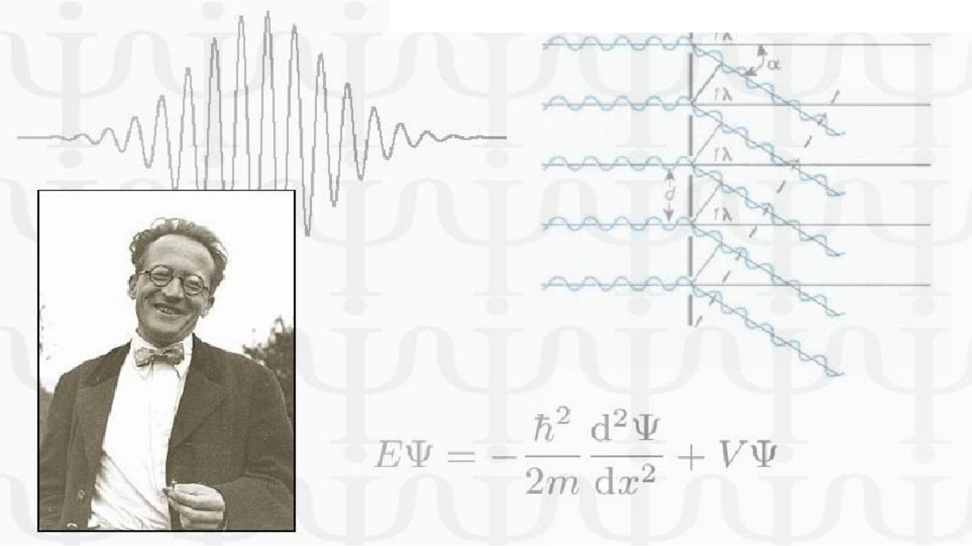
QUANTUM MECHANICS

UNIT II
Quantum Mechanics
Lecture-8





Existence of Protons, Neutrons, and α-particles in the Nucleus

- To prove the existence of protons, neutrons, and a-particles in the nucleus, let us start with the maximum uncertainty in the measurements of positions of these particles in the nucleus.

- This uncertainty will be equal to the order of the diameter of nucleus, i.e., $\Delta x = 10^{-14}$ m. Using the uncertainty principle, the uncertainty in the momentum of above said particles can be given as

$$\Delta p_x \geq \frac{h}{4\pi \cdot \Delta x}$$

$$= \frac{6.63 \times 10^{-34}\ \mathrm{Js}}{4 \times 3.14 \times (10^{-14}\ \mathrm{m})}$$

$$= 0.527 \times 10^{-20}\ \mathrm{kgm/s}$$

For protons and neutrons, $m_0 \simeq 1.67 \times 10^{-27}$ kg. This is a non-relativisitic problem as for these particles, $\upsilon = p/m_0 \simeq 3 \times 10^6$ m/s^{-1}. The kinetic energy in this case can be given as

$$E_k = \frac{p^2}{2m} = \frac{(0.527 \times 10^{-20})^2}{2 \times 1.67 \times 10^{-27}} \text{ J}$$

$$= \frac{(0.527 \times 10^{-20})^2}{2 \times 1.67 \times 10^{-27} \times 1.6 \times 10^{-19}} \text{ eV } = E_k = 52 \text{ keV}$$

This value of this kinetic energy is smaller than the energies of the particles emitted by the nucleus. Hence, the particles such as protons and neutrons, or the particles heavier than these, can exist inside the nucleus.

Radiation of Light from an Excited Atom

Since an atom takes an average time period of 10^{-8} s to come back to its ground state from the excited state, the uncertainty in the photon energy can be given as

$$\Delta E \geq \frac{h}{4\pi \cdot \Delta t} = \frac{6.6 \times 10^{-34}}{4 \times 3.14 \times 10^{-8}}$$

$$\simeq 0.525 \times 10^{-26} \text{ J}$$

Thus, the uncertainty in the frequency of the light is

$$\Delta \nu \geq \frac{\Delta E}{h/4\pi} = \frac{0.525 \times 10^{-26} \times 4 \times 3.14}{6.6 \times 10^{-34}} = 0.999 \times 10^{8} \text{ Hz}$$

$$\simeq 10^{8} \text{ Hz}$$

The above value of Δν gives the maximum limit of accuracy with which one can determine the frequency of the radiation emitted by an atom.

An electron has a speed of 1.05×10^4 m/s with an accuracy of 0.02%. Calculate the uncertainty in the position of the electron.

Solution

From the uncertainty principle, we have

$$\Delta x \cdot \Delta p = \hbar$$

$$p = mv = 9 \times 10^{-31} \times 1.05 \times 10^4$$

$$= 9.45 \times 10^{-27} \text{ kgm/s}$$

From the given problem, we have

$$\Delta p = \frac{0.02}{100} \times 9.45 \times 10^{-27}$$

$$= 18.9 \times 10^{-31}$$

$$= 1.89 \times 10^{-30}$$

Since, $\Delta x = \dfrac{\hbar}{\Delta p}$

$\Rightarrow$ $$\Delta x = \frac{6.63 \times 10^{-34}}{2 \times 3.14 \times 1.89 \times 10^{-30}}$$

$$= 0.558 \times 10^{-4} \text{ m}$$

$$= 5.58 \times 10^{-3} \text{ m}$$

Compare the uncertainties in the velocities of an electron and a proton confined in a box of equal dimensions. Their masses are 9.1×10^{-31} kg and 1.67×10^{-27} kg, respectively.

Solution

Since the electron and the proton are confined in a box of equal dimensions, the uncertainties in the positions of these particles will be the same. Let this uncertainty be Δx, i.e.,

$$\Delta x = \hbar/\Delta p_e \tag{1}$$

and

$$\Delta x = \hbar/\Delta p_p \tag{2}$$

where Δp_e and Δp_p are the uncertainties in the momentum of the electron and the proton, respectively.

On equating Eqs. (1) and (2), we get

$$\frac{\hbar}{\Delta p_e} = \frac{\hbar}{\Delta p_p}$$

Now,

$$\frac{\Delta p_p}{\Delta p_e} = \frac{1}{1}$$

or

$$\frac{\Delta \upsilon_p}{\Delta \upsilon_e} = \frac{m_e}{m_p} = \frac{9.1 \times 10^{-31}}{1.67 \times 10^{-27}}$$

or

$$\frac{m_p}{m_e} = \frac{\Delta \upsilon_e}{\Delta \upsilon_p} = \frac{1.67 \times 10^{-27}}{9.1 \times 10^{-31}}$$

or

$$\frac{\Delta \upsilon_e}{\Delta \upsilon_p} = 1835$$

A hydrogen atom is 0.53 Å in radius. Use uncertainty principle to estimate the minimum energy with which an electron can exist in this atom.

Solution

Uncertainty in the position of the electron is equal to the diameter of the hydrogen atom, i.e.,

$$\Delta x = 2 \times 0.53 \text{ Å}$$

$$\Delta x = 1.06 \times 10^{-10} \text{ m}$$

From the uncertainty principle, we have

$$\Delta x \cdot \Delta p \geq \hbar/2$$

$$\Delta p \geq \frac{6.63 \times 10^{-34}}{2 \times 2 \times 3.14 \times 1.06 \times 10^{-10}}$$

$$= 4.98 \times 10^{-25} \text{ kgm/s}$$

Now, the kinetic energy of electron is

$$E_k \geq \frac{p^2}{2m}$$

$$\geq \frac{(4.98 \times 10^{-25})^2}{2 \times 9.1 \times 10^{-31}}$$

$$\geq 1.363 \times 10^{-19} \text{ J}$$

$$\geq \frac{1.363 \times 10^{-19}}{1.6 \times 10^{-19}}$$

$$\geq 0.852 \text{ eV}$$

Note: When this problem is solved by using the uncertainty principle $\Delta x \cdot \Delta p_x = \hbar$, the minimum value of kinetic energy will come out to be 13.5 eV.

The position and the momentum of 0.5 keV electrons are simultaneously determined. If the position is located within 0.4 nm, what is the percentage of uncertainty in its momentum?

Solution

We have

$$E = 0.5 \times 10^3 \text{ eV}$$

$$= 0.5 \times 10^3 \times 1.6 \times 10^{-19} \text{ J}$$
$$= 8 \times 10^{-17} \text{ J}$$

Since it is a non-relativisitic case, the momentum of the electron can be given as

$$p = \sqrt{2mE}$$
$$= (2 \times 9 \times 10^{-31} \times 8 \times 10^{-17})^{1/2}$$
$$= 1.2 \times 10^{-23} \text{ kgm/s}$$

Uncertainty in the position can be given as

$$\Delta x = 0.4 \times 10^{-9} \text{ m}$$
$$= 4 \text{ Å}$$

From the uncertainty principle, we have

$$\Delta p = \frac{\hbar}{2 \cdot \Delta x}$$

$$= \frac{6.63 \times 10^{-34}}{2 \times 2 \times 3.14 \times 4 \times 10^{-10}}$$

$$\Delta p = 1.3 \times 10^{-25} \text{ kgm/s}$$

The percentage uncertainty in momentum can be given as

$$\frac{\Delta p}{p} \times 100 = \frac{1.3 \times 10^{-25}}{1.2 \times 10^{-23}} \times 100$$

$$= 1.08\%$$

Using uncertainty principle, show that the simultaneous measurements of position and momentum of a particle with absolute accuracy is impossible.

Solution

From the uncertainty principle, we have

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi}$$

Case I: When position is measured with absolute accuracy, then $\Delta x = 0$. Hence,

$$\Delta p = \frac{h}{4\pi(0)}$$
$$= \infty$$

This means that uncertainty in measuring the momentum (Δp) will be infinity, i.e., the velocity of the particle cannot be determined.

Case II: When momentum is measured with absolute accuracy, then $\Delta p = 0$. Hence,

$$\Delta x = \frac{h}{4\pi(0)}$$
$$= \infty$$

This means that the position of the particle cannot be determined.

Thus, both the momentum and the position of the particle cannot be determined simultaneously with absolute accuracy.

A baseball of mass 200 g is moving with a velocity of 6 m/s. If we can locate the baseball with an error equal to the magnitude of the wavelength of light used (5000 Å), then compare the uncertainty in the momentum with the total momentum of the baseball.

Solution

Step I: *Calculation of uncertainty in the momentum of the baseball.*

From the uncertainty principle, we have

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi}$$

or

$$\Delta p = \frac{h}{4\pi \times \Delta x}$$

Given that $\Delta x = 5000 \times 10^{-10}$ m $= 5 \times 10^{-7}$ m, $\pi = 3.14$, and $h = 6.63 \times 10^{-34}$ kgm/s.

Now, putting these values in Eq. (1), we get

$$\Delta p = \frac{6.63 \times 10^{-34}}{4 \times 3.14 \times 5 \times 10^{-7}}$$

$$= 1.055 \times 10^{-28} \text{ kgm/s}$$

Step II: *Calculation of* Δp

We know that

$$p = mv$$

$$= 200 \times 10^{-3} \times 6$$

$$= 1200 \times 10^{-3} \text{ kgm/s}$$

Now,

$$\frac{\Delta p}{p} = \frac{1.055 \times 10^{-28}}{1200 \times 10^{-3}}$$

$$= 8.79 \times 10^{-27}$$

An electron is bound in one-dimensional potential box which has the width 2.5×10^{-10} m. Assuming the height of the box to be infinite, calculate the two lowest permitted energy values of the electron.

Solution

The energy of a particle of mass m moving in one-dimensional potential box of infinite height and of width L is given as

$$E_n = \frac{n^2h^2}{8mL^2}$$

where $n = 1, 2, 3, \ldots$

The first lowest energy of the electron is obtained by putting $n = 1$ and the second lowest energy level corresponds to $n = 2$. Hence,

$$E_1 = \frac{h^2}{8mL^2}$$

and $$E_2 = 4 \cdot \frac{h^2}{8mL^2} = 4E_1$$

Now, $$E_1 = \frac{(6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (2.5 \times 10^{-10})^2}$$

$$= \frac{43.956 \times 10^{-68}}{455 \times 10^{-51}}$$

$$= 9.66 \times 10^{-19}\ \text{J}$$

$$= \frac{9.66 \times 10^{-19}}{1.6 \times 10^{-19}}\ \text{eV}$$

$$= 6.04\ \text{eV}$$

Since $E_2 = 4E_1$

$\Rightarrow$ $E_2 = 4 \times 6.04\ \text{eV} = 24.16\ \text{eV}$

Hence, the first two lowest energy levels of the electron are 6.04 eV and 24.16 eV, respectively.

Calculate the energy difference between the ground state and first excited state of an electron in a one-dimensional rigid box of length 10^{-8} cm.

Solution

The energy of a particle of mass m in a one-dimensional box of length L is given as

$$E_n = \frac{n^2 h^2}{8mL^2}$$

where $n = 1, 2, 3, ...$

$$E_n = \frac{n^2 (6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (10^{-10})^2} = 6.03 \times 10^{-18}\, n^2 \text{ J}$$

$$= \frac{6.03 \times 10^{-18}}{1.6 \times 10^{-19}} n^2 \text{ eV}$$

$$= 37.68\, n^2 \text{ eV}$$

For ground state, $n = 1$

Hence, $E_1 = 37.68$ eV

For first excited state, $n = 2$

Hence, $E_2 = 37.68 \times 2^2 = 150.72$ eV

Difference in energy between the first excited state and the ground state can be given as

$$\Delta E = E_2 - E_1 = 150.72 - 37.68$$

$$= 113.04 \text{ eV}$$

An electron is confined to move between two rigid walls separated by 1 Å. Find the de Broglie wavelength representing the first three allowed energy states of the electron and their corresponding energies.

Solution

The backward and forward motions of the electron between the rigid walls of the box form a stationary wave pattern with nodes at the walls. Hence, the distance between the walls will be a whole multiple of half of the de Broglie wavelength. Hence, we get

$$L = n\left(\frac{\lambda}{2}\right)$$

where $n = 1, 2, 3, ...$

or $$\lambda = \left(\frac{2L}{n}\right)$$

Given that $L = 1\ \text{Å} = 10^{-10}$ m

Thus, corresponding to $n = 1, 2, 3, ...$,

$$\lambda = \frac{2 \times 1\text{Å}}{n} = 2\ \text{Å},\ 1\ \text{Å},\ 0.667\ \text{Å} ...$$

The energy of the electron in the box can be given as

$$E_n = \frac{n^2 h^2}{8mL^2}$$

$$= \frac{n^2 \times (6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31}\ (10^{-10})^2}$$

$$= 6.04 \times 10^{-18}\ n^2\ \text{J}$$

$$= \frac{6.04 \times 10^{-18}}{1.6 \times 10^{-19}} n^2\ \text{eV}$$

$$= n^2 \cdot 37.73\ \text{eV}$$

$E_1 = 37.73$ eV (For $n = 1$)

$E_2 = 150.92$ eV (For $n = 2$)

$E_3 = 339.57$ eV (For $n = 3$)

A particle is moving in a one-dimensional box of width 30 Å. Calculate the probability of finding the particle within and at interval of 2 Å at the centre of the box when it is in its state of least energy.

Solution

The wave function of a particle trapped in a deep potential box is given as

$$\psi_n(x) = \sqrt{\left(\frac{2}{L}\right)} \sin \frac{n\pi x}{L}$$

In the ground state ($n = 1$), particle will be at its lowest energy state. Therefore,

$$\psi(x) = \sqrt{\left(\frac{2}{L}\right)} \sin \frac{\pi x}{L}$$

At the centre of the box, $x = L/2$. The probability of finding the particle in the unit interval at the centre of the box is given as

$$|\psi(x)|^2 = \left[\sqrt{\left(\frac{2}{L}\right)} \sin \frac{\pi(L/2)}{L}\right]^2$$

$$= \frac{2}{L} \sin^2 \frac{\pi}{2} = \frac{2}{L}$$

The probability P in the interval Δx is given as

$$P = |\psi(x)|^2 \Delta x$$

$$P = \frac{2}{L} \Delta x$$

Given that $L = 30$ Å and $\Delta x = 2$ Å

$$\Rightarrow \qquad P = \frac{2}{30} \times 2$$

$$= 0.16$$

$$= 16\ \%$$