Control Systems

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Unit-II

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Lecture 2

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The state variables describe the future response of a system, given the present state, the excitation inputs, and the equations describing the dynamics.

A simple example of a state variable is the state of an on-off light switch. The switch can be in either the on or the off position, and thus the state of the switch can assume one of two possible values. Thus, if we know the present state (position) of the switch at t_0 and if an input is applied, we are able to determine the future value of the state of the element.

The concept of a set of state variables that represent a dynamic system can be illustrated in terms of the spring-mass-damper system shown in Figure 2. The number of state variables chosen to represent this system should be as small as possible in order to avoid redundant state variables. A set of state variables sufficient to describe this system includes the position and the velocity of the mass.

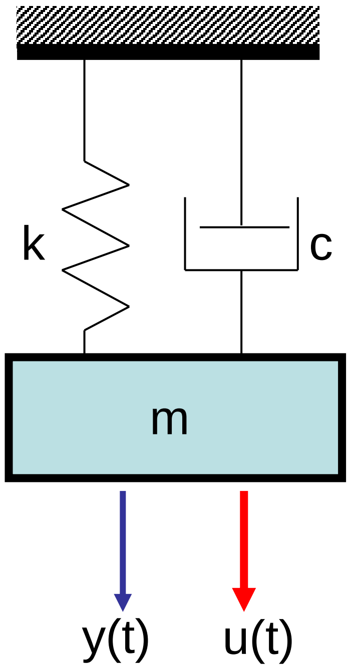

Figure 2. 1-dof system.

Therefore we will define a set of variables as [x_1 x_2], where

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt}$$

Kinetic and Potential energies, virtual work.

$$E_1 = \frac{1}{2} m \dot{y}^2, \quad E_2 = \frac{1}{2} k y^2, \quad \delta W = u(t)\,\delta y - c\,\dot{y}\,\delta y$$

Lagrangian of the system is expressed as $L = E_1 - E_2$

Generalized Force

Lagrange's equation $$\frac{d}{dt}\left(\frac{\partial(E_1 - E_2)}{\partial \dot{y}}\right) - \frac{\partial(E_1 - E_2)}{\partial y} = Q_y$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + k\,y = u(t)$$

$$m\frac{dx_2}{dt} + c\,x_2 + k\,x_1 = u(t)$$

Equation of motion in terms of state variables.

We can write the equations that describe the behavior of the spring-mass-damper system as the set of two first-order differential equations.

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u(t)$$

This set of difefrential equations describes the behavior of the state of the system in terms of the rate of change of each state variables.

As another example of the state variable characterization of a system, consider the RLC circuit shown in Figure 3.

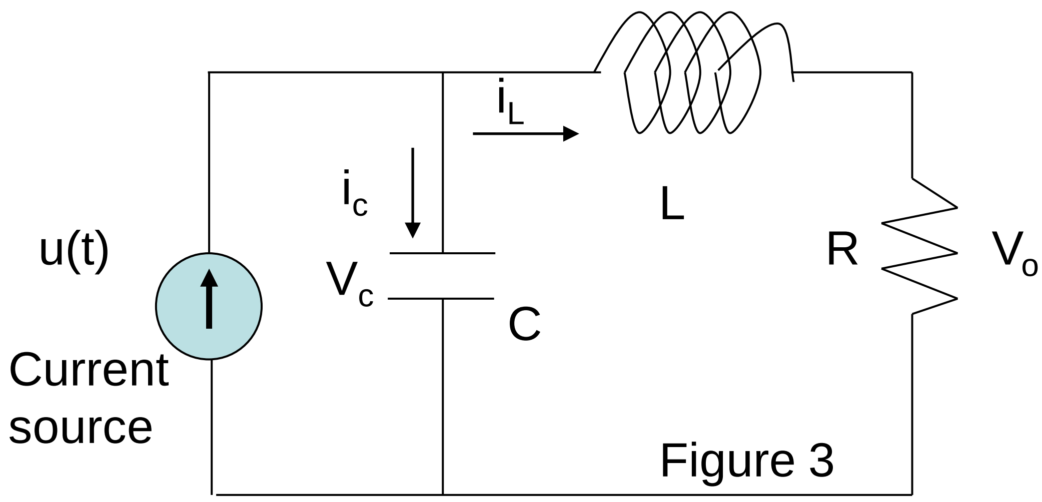

Figure 3

The state of this system can be described in terms of a set of variables [x_1 x_2], where x_1 is the capacitor voltage $v_c(t)$ and x_2 is equal to the inductor current $i_L(t)$. This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables.

$$E_1 = \frac{1}{2}L\,i_L^{\,2}, \quad E_2 = \frac{1}{2C}\left(\int i_c dt\right)^2 = \frac{1}{2}C\,v_c^2$$

Therefore $x_1(t_0)$ and $x_2(t_0)$ represent the total initial energy of the network and thus the state of the system at $t=t_0$.

Utilizing Kirchhoff’s current low at the junction, we obtain a first order differential equation by describing the rate of change of capacitor voltage

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L$$

Kirchhoff’s voltage low for the right-hand loop provides the equation describing the rate of change of inducator current as

$$L\frac{di_L}{dt} = -R\,i_L + v_c$$

The output of the system is represented by the linear algebraic equation

$$v_0 = R\,i_L(t)$$

We can write the equations as a set of two first order differential equations in terms of the state variables x_1 [$v_C(t)$] and x_2 [$i_L(t)$] as follows:

$$C\frac{dv_c}{dt} = u(t) - i_L \quad \Longrightarrow \quad \frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$

$$L\frac{di_L}{dt} = -R\,i_L + v_c \quad \Longrightarrow \quad \frac{dx_2}{dt} = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

The output signal is then $y_1(t) = v_0(t) = R\,x_2$

Utilizing the first-order differential equations and the initial conditions of the network represented by [$x_1(t_0)$ $x_2(t_0)$], we can determine the system's future and its output.

The state variables that describe a system are not a unique set, and several alternative sets of state variables can be chosen. For the RLC circuit, we might choose the set of state variables as the two voltages, $v_C(t)$ and $v_L(t)$.

In an actual system, there are several choices of a set of state variables that specify the *energy stored in a system* and therefore adequately describe the dynamics of the system.

The state variables of a system characterize the dynamic behavior of a system. The engineer's interest is primarily in physical, where the variables are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables.

The State Differential Equation:

The state of a system is described by the set of first-order differential equations written in terms of the state variables [x_1 x_2 ... x_n]. These first-order differential equations can be written in general form as

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n + b_{11}u_1 + \cdots b_{1m}u_m$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n + b_{21}u_1 + \cdots b_{2m}u_m$$
$$\vdots$$
$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n + b_{n1}u_1 + \cdots b_{nm}u_m$$

Thus, this set of simultaneous differential equations can be written in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

n: number of state variables, m: number of inputs.

The column matrix consisting of the state variables is called the **state vector** and is written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The vector of input signals is defined as u. Then the system can be represented by the compact notation of the state differential equation as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

This differential equation is also commonly called the state equation. The matrix **A** is an nxn square matrix, and **B** is an nxm matrix. The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Where **y** is the set of output signals expressed in column vector form. The state-space representation (or state-variable representation) is comprised of the state variable differential equation and the output equation.

- A(t) is called the state matrix,
- B(t) the input matrix,
- C(t) the output matrix, and
- D(t) the direct transmission matrix.