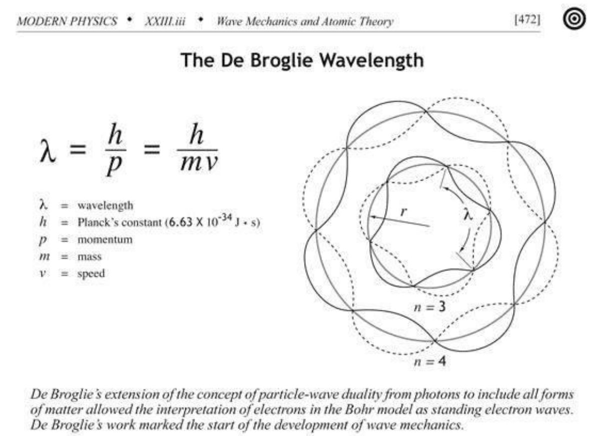
QUANTUM MECHANICS

UNIT II
Quantum Mechanics
Lecture-4

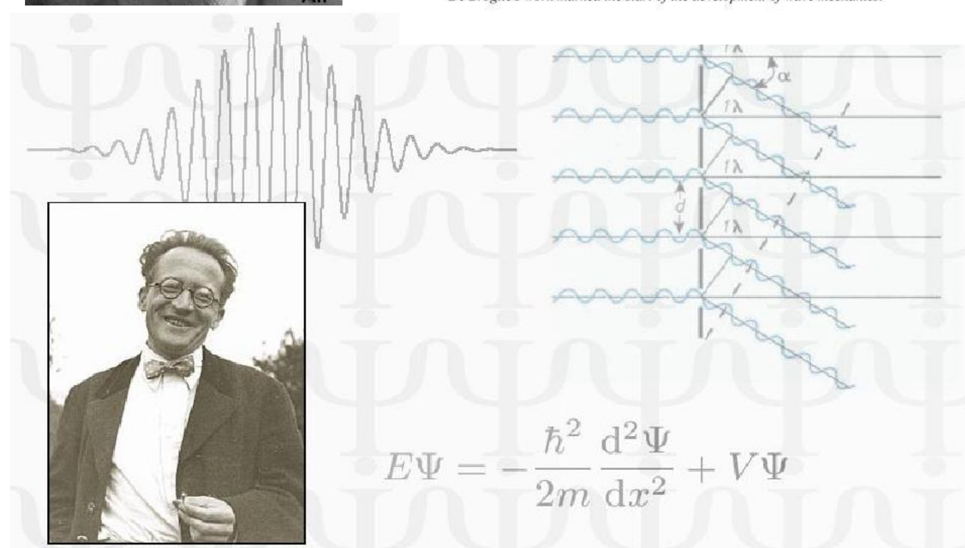

MODERN PHYSICS • XXIII.iii • Wave Mechanics and Atomic Theory [472]

The De Broglie Wavelength

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

λ = wavelength
h = Planck's constant (6.63×10^{-34} J • s)
p = momentum
m = mass
v = speed

De Broglie's extension of the concept of particle-wave duality from photons to include all forms of matter allowed the interpretation of electrons in the Bohr model as standing electron waves. De Broglie's work marked the start of the development of wave mechanics.



BASIS FOR UNCERTAINTY PRINCIPLE

✓Although in the beginning scientists were reluctant to accept this principle, but the strong evidences forced them to accept the uncertainty principle.

BASIS FOR UNCERTAINTY PRINCIPLE

- The material particle exhibits particle nature as well as exhibits wave nature, but it does not simultaneously possess both the natures.

- Instead of being contradictory, the wave and particle natures are complementary.

- Bohr's principle of complementarity is the consequence of de Broglie hypothesis.

- Under the de Broglie hypothesis, particles may be represented as wave packets. The particle may be anywhere inside the wave packet. Hence, there will be uncertainty in the measurement of position of the particle.

HEISENBERG'S UNCERTAINTY PRINCIPLE

- The Heisenberg's uncertainty principle states that it is not possible to simultaneously measure the position and the momentum of a particle to any desired degree of accuracy.

- In other words, the product of uncertainty in the measurement of position (Δx) and uncertainty in the measurement of momentum (Δp) is always constant, and it is at least equal to Planck's constant (h), i.e.,

$$\Delta p \cdot \Delta x = h$$

Similar to above expression, we can write

$$\Delta E \cdot \Delta t = h$$

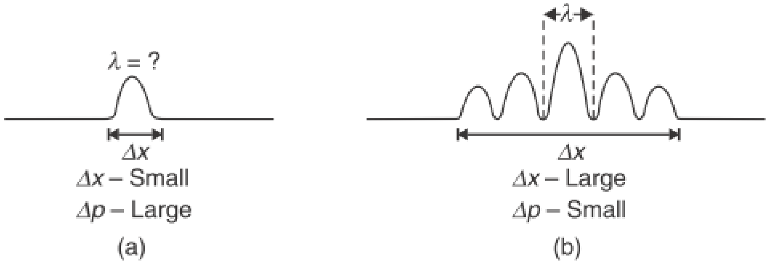
and

$$\Delta J \cdot \Delta \theta = h$$

where ΔE and Δt are the uncertainties in determining energy and time, respectively. Similarly, ΔJ and $\Delta \theta$ are the uncertainties in the measurement of angular momentum and angle, respectively.

Explanation

- To understand the uncertainty in the measurement of position and momentum of microscopic particles, let us take the examples of narrow and wide wave packets.



- In a narrow wave packet [Fig.(a)], the position of the particle can be precisely determined, but not the wavelength.

- As a result, the particle's momentum cannot be measured accurately as there are not enough waves to exactly measure the wavelength ($\lambda = h/mv$).

- On the other hand, in a wider wave packet [Fig.(b)], the wavelength can be determined exactly but the position of the particle will be uncertain due to the large width of the wave packet.

- Hence, it can be concluded that it is impossible to simultaneously determine the exact position and the exact momentum of a particle.

DERIVATION OF UNCERTAINTY PRINCIPLE

- In order to derive uncertainty principle let us consider two simple harmonic plane waves of same amplitude A having nearly equal frequencies w_1 and w_2 with propagation vectors k_1 and k_2, respectively.

$$y_1 = A \sin(\omega_1 t - k_1 x)$$

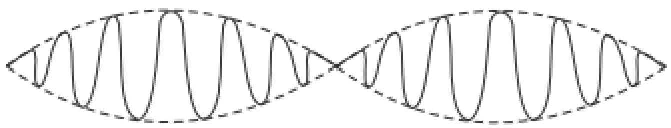
$$y_2 = A \sin(\omega_2 t - k_2 x)$$

Using the principle of superposition the resultant equation can be given as

$$2A \sin(\omega t - kx) \cos\left(\frac{\Delta\omega}{2} t - \frac{\Delta k}{2} x\right)$$

DERIVATION OF UNCERTAINTY PRINCIPLE

The resultant of these equations can be given as the wave packet given below



In wave packets, the position of the particle remains uncertain between successive nodes

$$\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) = 0$$

Now, $$X_{n+1} - X_n = \Delta x = \frac{2\pi}{\Delta k}$$

DERIVATION OF UNCERTAINTY PRINCIPLE

or $$\Delta k = \frac{2\pi\Delta p}{h}$$

or $$\Delta p = \frac{\Delta k \cdot h}{2\pi}$$

Now, from Eqs. (23.11) and (23.12), we get

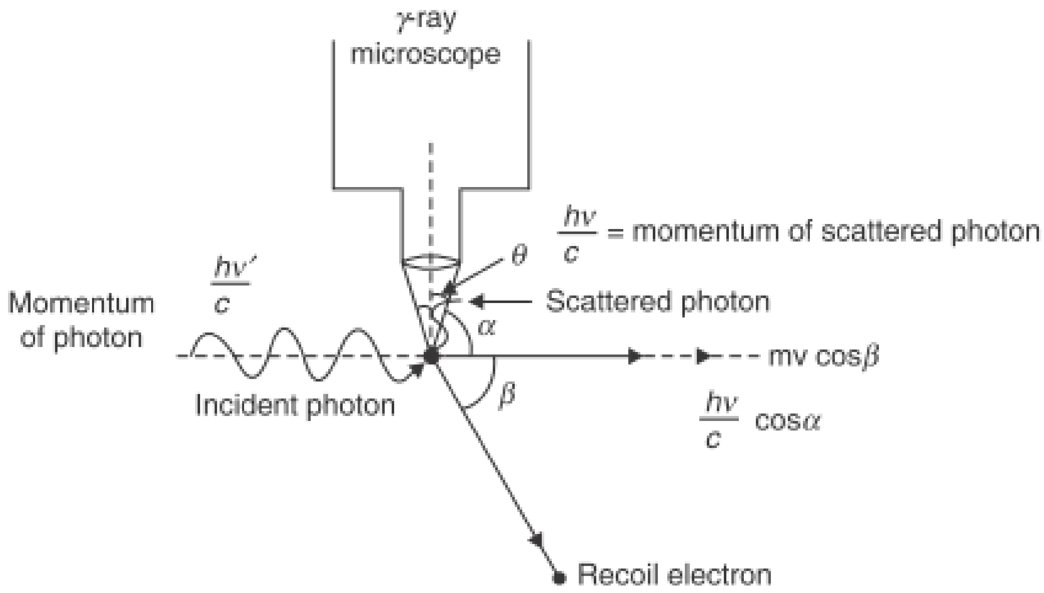
$$\Delta x \cdot \Delta p = \frac{\cancel{2}\pi}{\cancel{\Delta}k} \cdot \frac{\cancel{\Delta}k \cdot h}{\cancel{2}\pi}$$

$$\Delta x \cdot \Delta p = h$$

EXPERIMENTAL EXAMPLES OF UNCERTAINTY PRINCIPLE

Determination of the Position of a Particle by γ-ray Microscope

To measure the exact position and the momentum of an electron along the X-axis in the field of view of an ideally high resolving power microscope, let us consider a photon being incident on an electron in the field of view of microscope as shown in Fig.



EXPERIMENTAL EXAMPLES OF UNCERTAINTY PRINCIPLE

The resolving power of a microscope can be given as

$$\Delta x = \frac{\lambda}{2\sin\theta}$$

and

$$\frac{h}{c}(\nu' - \nu\cos(90^\circ - \theta)) \leq p_x \leq \frac{h}{c}(\nu' - \nu\cos(90^\circ + \theta))$$

$$\Delta p_x = \frac{h}{c}\,[\nu' - \nu\cos(90^\circ + \theta) - \nu' + \nu\cos(90^\circ - \theta)]$$

$$= \frac{h}{c}\,(\nu\sin\theta + \nu\sin\theta)$$

$$= \frac{2h\nu}{c}\,\sin\theta$$

$$\Delta p_x = \frac{2h\sin\theta}{\lambda}$$

Combining above expressions of Δp and Δx we get

$$\Delta x \cdot \Delta p_x = h$$

TIME–ENERGY UNCERTAINTY PRINCIPLE

- We can derive the expression for time–energy uncertainty with the help of position and momentum uncertainties.
- Let us consider a particle of rest mass m_0 moving with velocity v_x in the X-direction. The kinetic energy of the particle can be given as

$$E = \frac{1}{2} m_0 v_x^2 = \frac{p_x^2}{2m_0}$$

TIME–ENERGY UNCERTAINTY PRINCIPLE

$$\Delta E = \frac{2p_x \cdot \Delta p_x}{2m_0}$$

If ΔE and Δp_x are the uncertainties in energy and momentum, respectively, then

$$p_x \cdot \Delta p_x = m_0 \, \Delta E$$

or $$\Delta p_x = \frac{m_0}{p_x} \cdot \Delta E = \frac{1}{\upsilon_x} \cdot \Delta E \qquad (23.15)$$

Let the uncertainty in measuring the time interval at point x be Δt. Then, uncertainty in position Δx can be given as

$$\Delta = \upsilon_x \cdot \Delta t \qquad (23.16)$$

From Eqs. (23.15) and (23.16), we get

$$\Delta p_x \cdot \Delta x = \frac{1}{\upsilon_x} \Delta E \cdot \upsilon_x \cdot \Delta t$$
$$= \Delta E \cdot \Delta t$$

or $$\Delta E \cdot \Delta t = h \qquad [(\text{from Eq. (23.1)}, \Delta x \cdot \Delta p_x = h)]$$

or $$\Delta E \cdot \Delta t \geq h/4\pi \qquad (23.17)$$

Applications of Uncertainty Principle

1. Non-Existence of Electrons in the Nucleus

- Since the diameter of nucleus is of the order of 10^{-14} m, the maximum uncertainty in the measurement of position of the electron in the nucleus will be of the order of $\Delta x = 10^{-14}$ m.

- Using Heisenberg's uncertainty relation, the uncertainty in the measurement of moment of the electron is given as

$$\Delta p_x \geq \frac{h}{4\pi \cdot \Delta x}$$

$$= \frac{6.63 \times 10^{-34}\ \text{Js}}{4 \times 3.14 \times (10^{-14}\ \text{m})} \quad (\Delta x = 10^{-14}\ \text{m})$$

$$= 0.527 \times 10^{-20}\ \text{kgm/s}$$

or

$$\Delta p_x \geq 0.527 \times 10^{-20}\ \text{kgm/s}$$

Applications of Uncertainty Principle

$$E = pc$$

$$\simeq (0.527 \times 10^{-20}) \times (3 \times 10^{8})$$

$$= \frac{0.527 \times 10^{-20} \times 3 \times 10^{8}}{1.6 \times 10^{-19}} \text{ eV}$$

$$E = 9.88 \text{ MeV}$$

- The above calculation shows that an electron can exist in the nucleus if its energy is of the order of 9.88 MeV.

- But we know that the electrons emitted by radioactive nuclei during β-decay have energies of the order of 3 MeV to 4 MeV only.

- Hence, electrons cannot exist in the nucleus.

Zero-Point Energy of a Harmonic Oscillator

- From quantum mechanics, we know that the lowest energy of a simple harmonic oscillator is not zero; instead it is equal to 1/2 ħ w (where ħ = h/2π) and is known as zero-point energy.
- This zero-point energy of the oscillator can be obtained with the help of uncertainty principle
- Δx and Δp_x be the uncertainties in the simultaneous measurements of the position and the momentum of a particle of mass m executing simple harmonic motion along the X-axis.
- Now, from the uncertainty principle, we can write

$$\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}$$

or $$\Delta p_x = \frac{\hbar}{2\Delta x}$$

Total energy of the particle of mass m can be given as

$$E = E_K + V$$

$$= \frac{(\Delta p_x)^2}{2m} + \frac{1}{2}k(\Delta x)^2$$

where k is the force constant.

Now, putting the value of Δp_x in the above equation, we get

$$E = \frac{(\hbar/2\Delta x)^2}{2m} + \frac{1}{2}k(\Delta x)^2$$

$$= \frac{\hbar}{8m(\Delta x)^2} + \frac{1}{2}k(\Delta x)^2$$

Using the condition of minima, i.e., $\frac{dE}{d(\Delta x)} = 0$, we get

$$\Delta x = \left(\frac{\hbar^2}{4mk}\right)^{1/4}$$

Using this value of Δx in the total energy expression, we get

$$E_{\min} = \frac{\hbar^2}{8m}\left(\frac{4mk}{\hbar^2}\right)^{1/2} + \frac{1}{2}k\left(\frac{\hbar^2}{4mk}\right)^{1/2}$$

$$= \frac{\hbar}{4m}\left(\frac{k}{m}\right)^{1/2} + \frac{\hbar}{4}\left(\frac{k}{m}\right)^{1/2}$$

$$= \frac{\hbar}{2}\left(\frac{k}{m}\right)^{1/2}$$

Since $\left(\frac{k}{m}\right)^{1/2} = \omega$ (angular velocity), the $E_{\min}$ of the harmonic oscillator can be given as

$$E_{\min} = E_0 = \frac{1}{2}\hbar\omega$$

This relation is called the zero-point energy of a simple harmonic oscillator.

APPLICATIONS OF UNCERTAINTY PRINCIPLE CONTD…

Some other applications of uncertainty principle can be given as

- Existence of Protons, Neutrons, and α-particles in the Nucleus can be proved with the use of uncertainty principle.
- Binding Energy of an Electron in an Atom can be calculated with help of uncertainty principle
- Radiation of Light emitted from an Excited Atom can be calculated with help of uncertainty principle

Consequences of Uncertainty Principle

- The most important consequence of uncertainty principle is the dual nature of matter.
- In the dual nature, it is not possible to determine the wave and particle properties exactly at the same time.
- The complementarity principle states that the wave and particle aspects of matter are complementary, instead of being contradictory.
- This principle suggests that the consideration of particle and light natures is necessary to have a complete picture of the same system.

Example-1

An electron microscope is used to locate an electron in an atom within a distance of 0.2 Å. What is the uncertainty in the momentum of the electron located in this way?

Solution

From the Heisenberg's uncertainty principle, we have

$$\Delta x \cdot \Delta p \geq \hbar/2$$

or $$\Delta p = \frac{\hbar}{2\Delta x}$$

where Δp and Δx are the uncertainties in momentum and position, respectively.

Given that $\Delta x = 0.2 \text{ Å} = 0.2 \times 10^{-10}$ m

and $h = 6.63 \times 10^{-34}$ Js

$$\therefore \quad \Delta p = \frac{h}{2\pi \times 2 \times \Delta x}$$

$$= \frac{6.63 \times 10^{-34}}{2 \times 3.14 \times 2 \times 0.2 \times 10^{-10}}$$

$$= 2.64 \times 10^{-24} \text{ kgm/s}$$

Example-2

Calculate the smallest possible uncertainty in the position of an electron moving with a velocity of 3×10^7 m/s.

Solution

From the principle of unc

$\Delta x \cdot \Delta p \geq$

or $\Delta x = \frac{}{4\pi}$

Given that $\upsilon = 3 \times 10^7$

Now, $\Delta p \simeq p =$

Using the value of Δp i

$\Delta x = \frac{0.5}{}$

$$\Delta x = \frac{0.528 \times 10^{-34} \times \sqrt{1 - \left(\frac{3 \times 10^7}{3 \times 10^8}\right)^2}}{9 \times 10^{-31} \times 3 \times 10^7}$$

$$= \frac{0.528 \times 10^{-34} \times 0.995}{27 \times 10^{-24}}$$

$$= \frac{0.01945 \times 10^{-34}}{10^{-24}}$$

$$= 1.94 \times 10^{-12} \text{ m}$$

$$= 0.0194 \text{ Å}$$

Here $\upsilon = 3 \times 10^7$ m/s, $c = 3 \times 10^8$ m/s, and $m_0 = 9 \times 10^{-31}$ kg. Using these values in the above equation, we get

Example-3

An electron has the velocity of 600 m/s with an accuracy of 0.005%. Calculate the uncertainty with which we can locate the position of the electron.

Solution

Uncertainty in the velocity can be given as

$$\Delta \upsilon = 600 \times \frac{0.005}{100}$$
$$= 0.030 \text{ m/s}$$

Now, uncertainty in momentum can be given as

$$\Delta p_x = (9.1 \times 10^{-31}) \times 0.030$$
$$= 2.73 \times 10^{-32} \text{ kgm/s}$$

From the uncertainty principle, we have

$$\Delta x = \frac{\hbar}{\Delta p_x} = \frac{6.63 \times 10^{-34}}{2 \times 3.14 \times 2.73 \times 10^{-32}}$$
$$= 0.39 \times 10^{-2} \text{ m}$$

Assignment Based on this Lecture

- Describe the basis of uncertainty principle.
- Heisenberg uncertainty principle.
- Obtain the expression of uncertainty principle for position and momentum.
- Discuss experimental examples of uncertainty principle
- Explain the consequences of Uncertainty principle.
- Proof of Non existence of electron in the nucleus.
- Other applications of Uncertainty Principle.