Impulse response of first and second order systems

First order system $H(s)=\frac{K\tau}{\tau s+1} \Rightarrow h(t)=Ke^{-\frac{t}{\tau}}$

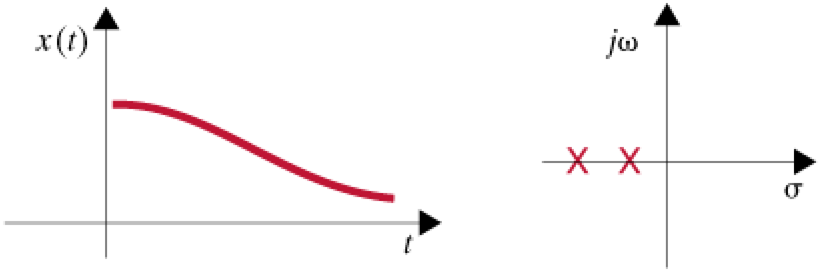
Normalized second order system

$$H(s)=\frac{\omega_0^2}{s^2+2\varsigma\omega_0 s+\omega_0^2}$$

poles: $s_{1,2}=-\varsigma\omega_0 \pm \omega_0\sqrt{\varsigma^2-1}$

Case 1: $\varsigma>1$: Overdamped network

$$h(t)=K_1e^{-(\varsigma\omega_0+\omega_0\sqrt{\varsigma^2-1})t}+K_2e^{-(\varsigma\omega_0-\omega_0\sqrt{\varsigma^2-1})t}$$

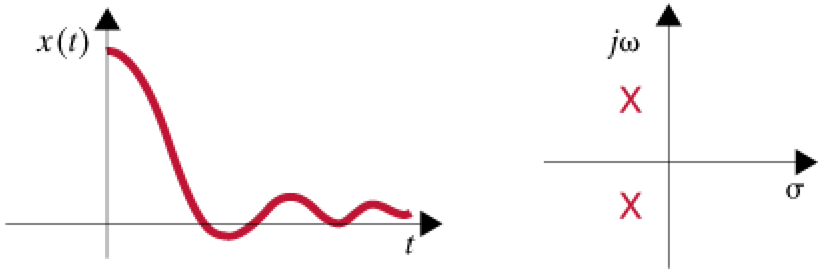


(a)

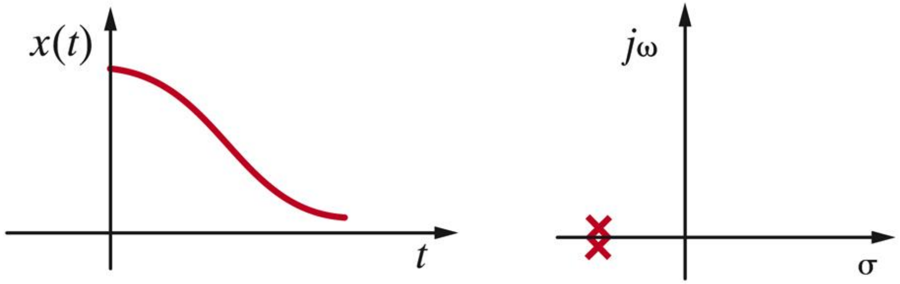
Case 2: $\varsigma<1$: Underdamped network

poles: $s_{1,2}=-\varsigma\omega_0 \pm j\omega_0\sqrt{1-\varsigma^2}$

$$h(t)=Ke^{-\varsigma\omega_o t}\cos(\omega_o\sqrt{1-\varsigma^2}t+\phi)$$



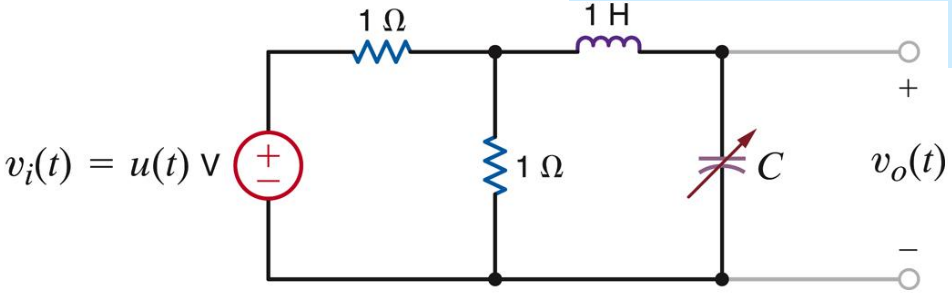
Case 3: $\varsigma=1$: Critically damped network



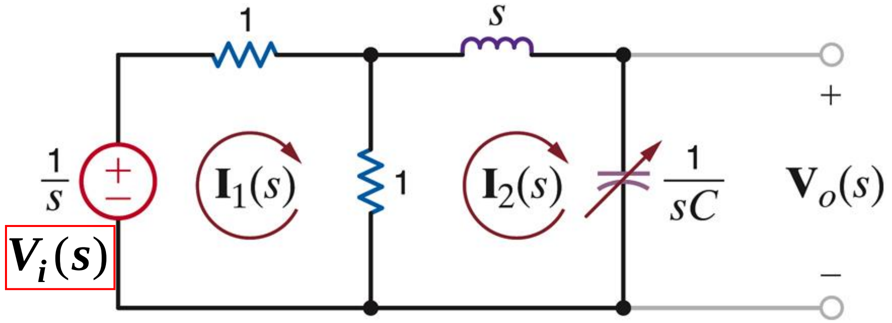
◀ ▶
GEAUX

LEARNING EXAMPLE

Determine the transfer function $H(s) = \frac{V_o(s)}{V_i(s)}$



Transform the circuit to the Laplace domain. All initial conditions set to zero



Mesh analysis

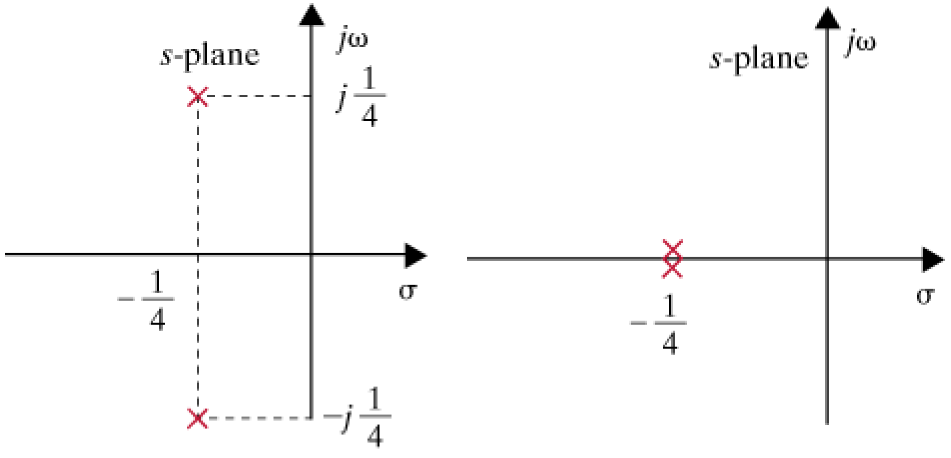
$$V_i(s) = 2I_1 - I_2$$

$$0 = -I_1 + \left(1 + s + \frac{1}{sC}\right) I_2$$

$$V_o(s) = \frac{1}{sC} I_2(s)$$

$$V_o(s) = \frac{(1/2C)}{s^2 + (1/2)s + 1/C}$$

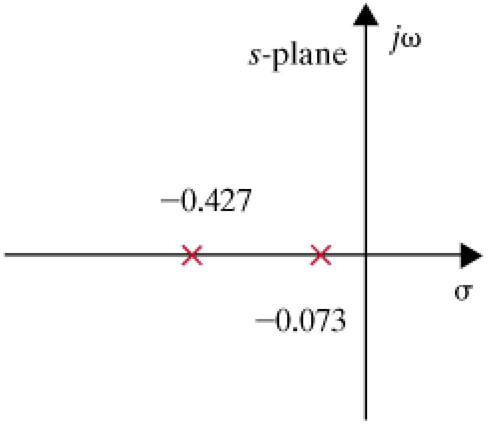
a) C = 8F ⇒ poles: $s_{1,2} = -0.25 \pm j0.25$



(c)

b) C = 16F ⇒ poles: $s_{1,2} = -0.25$

c) C = 32F ⇒ poles: $s_{1,2} = -0.427, -0.073$



(e)

GEAUX

LEARNING EXTENSION

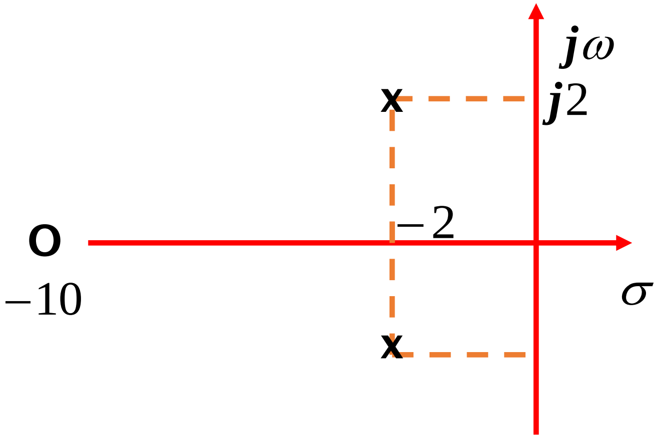
Determine the pole-zero plot, the type of damping and the unit step response

$$H(s)=\frac{s+10}{s^2+4s+8}$$

zero: $z=-10$

poles:

$$s^2+4s+8=0 \Rightarrow s_{1,2}=-2\pm j2$$



$$s^2+\boxed{4}s+\boxed{8}$$

$$\boxed{2\varsigma\omega_o} \quad \boxed{\omega_o^2}$$

$$\Rightarrow \varsigma=\frac{\sqrt{2}}{2}$$

$$Y(s)=H(s)\frac{1}{s}=\frac{s+10}{s(s^2+4s+8)}$$

$$s^2+4s+8=(s+2-j2)(s+2+j2)$$

$$Y(s)=\frac{K_1}{s}+\frac{K_2}{s+2-j2}+\frac{K_2^*}{s+2+j2}$$

$$\frac{K_1}{(s+\alpha-j\beta)}+\frac{K_1^*}{(s+\alpha+j\beta)} \leftrightarrow 2|K_1|e^{-\alpha t}\cos(\beta t+\angle K_1)u(t)$$

$$K_1=sY(s)|_{s=0}=\frac{10}{8}$$

$$K_2=(s+2-j2)V_o(s)|_{s=-2+j2}=\frac{8+j2}{(-2+j2)(j4)}$$

$$K_2=\frac{8.25\angle 14}{2.83\angle 135^\circ\times 4\angle 90^\circ}=0.73\angle-211^\circ$$

$$v_o(t)=\left(\frac{10}{8}+1.46e^{-2t}\cos(2t-211^\circ)\right)u(t)$$

GEAUX

Second order networks: variation of poles with damping ratio

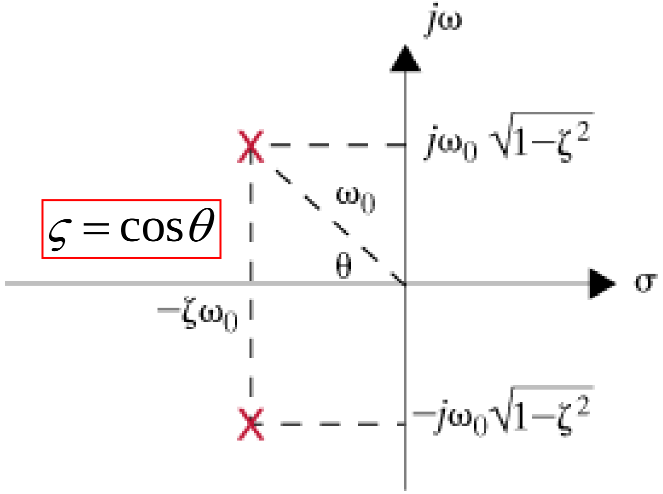
Normalized second order system

$$H(s)=\frac{\omega_0^2}{s^2+2\varsigma\omega_0 s+\omega_0^2}$$

$$\text{poles}: \quad s_{1,2}=-\varsigma\omega_0\pm\omega_0\sqrt{\varsigma^2-1}$$

Case 2: $\varsigma<1$: Underdamped network

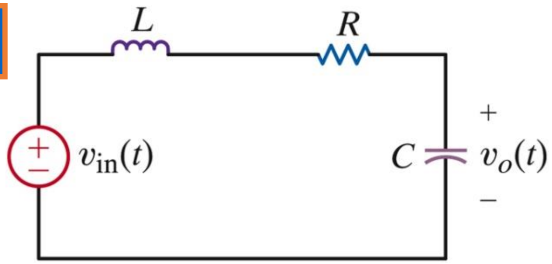
$$\text{poles}: \quad s_{1,2}=-\varsigma\omega_0\pm j\omega_0\sqrt{1-\varsigma^2}$$



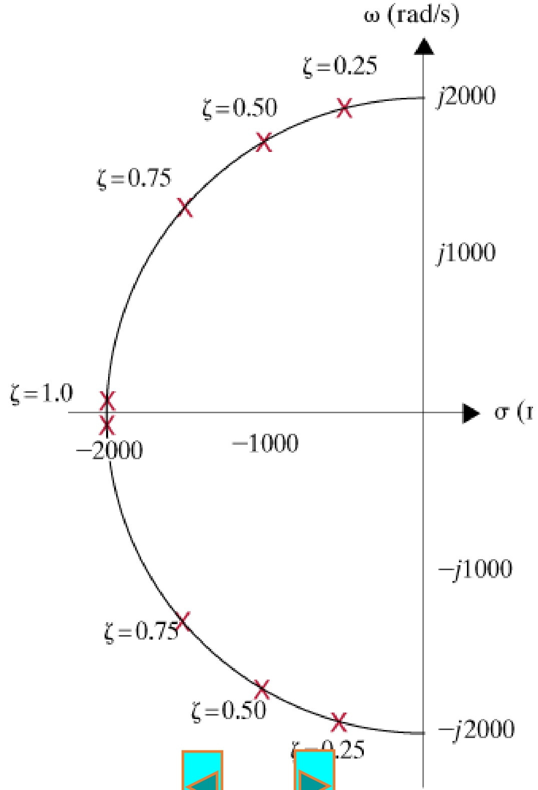
$$\boxed{\varsigma=\cos\theta}$$

LEARNING EXAMPLE

$$\omega_o^2=\frac{1}{LC}, \qquad 2\varsigma\omega_o=\frac{R}{L}$$

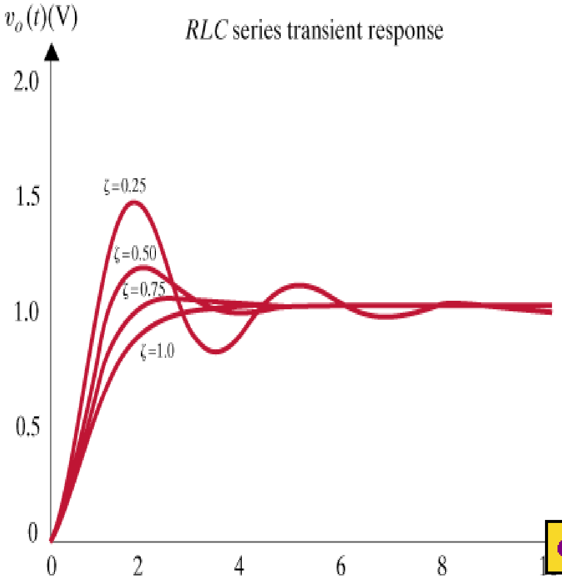


$$G_v(s)=\frac{V_o(s)}{V_{in}(s)}=\frac{\frac{1}{Cs}}{\frac{1}{Cs}+Ls+R}=\frac{\frac{1}{LC}}{s^2+\left(\frac{R}{L}\right)s+\frac{1}{LC}}$$



Variation of poles.

Use $\omega_o=2000$



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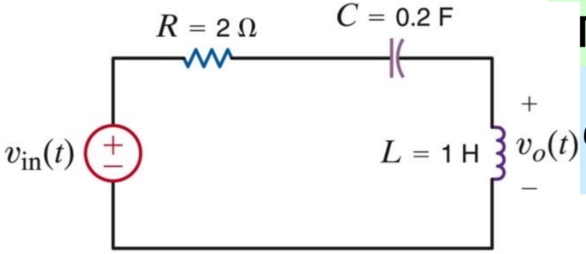
POLE-ZERO PLOT/BODE PLOT CONNECTION

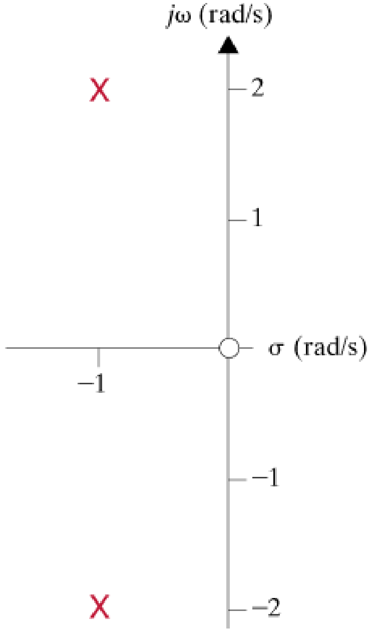
Bode plots display magnitude and phase information of $G(s)|_{s=j\omega}$

They show a cross section of G(s)

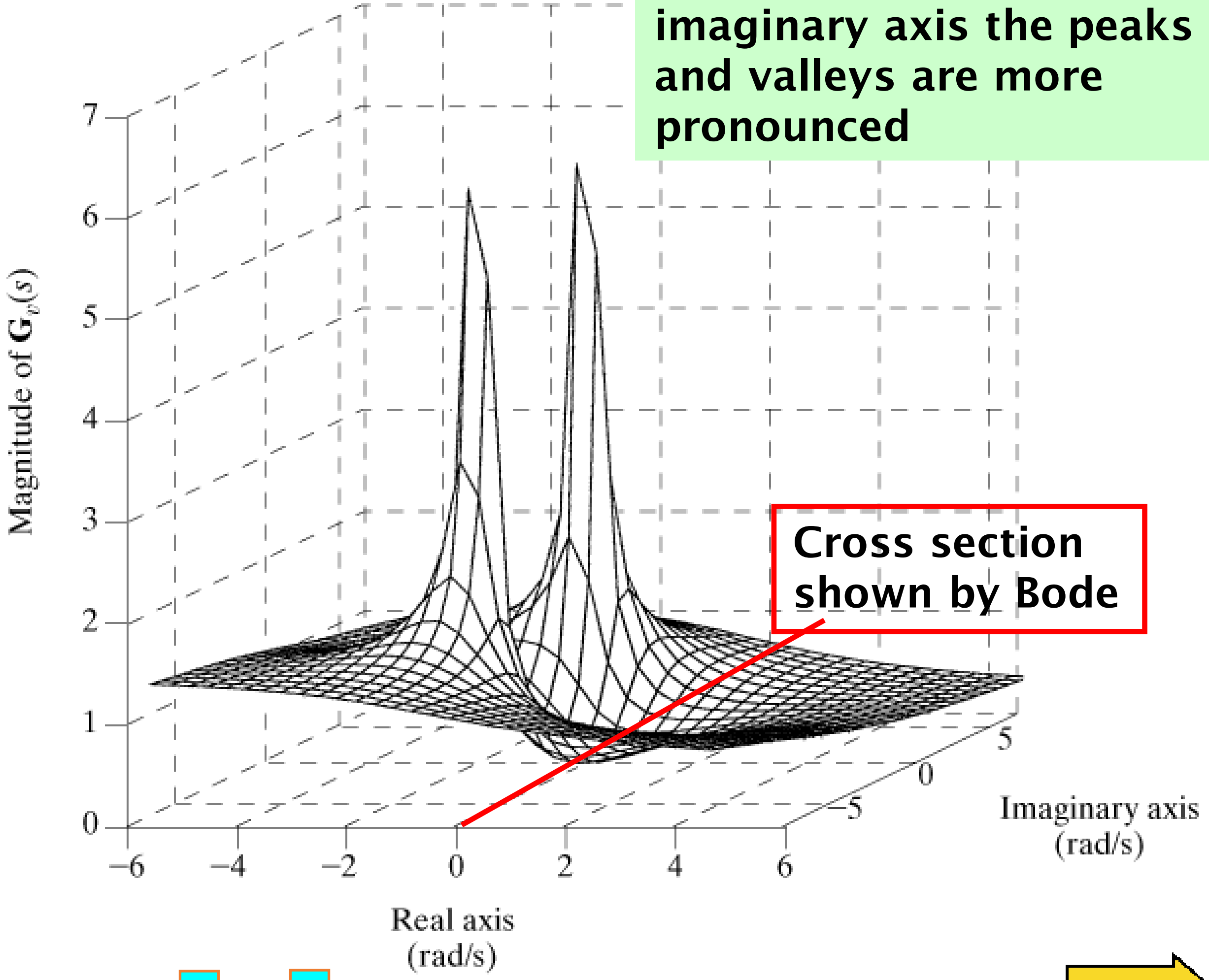


$$G(s)=\frac{s^2}{s^2+2s+5}$$

If the poles get closer to imaginary axis the peaks and valleys are more pronounced



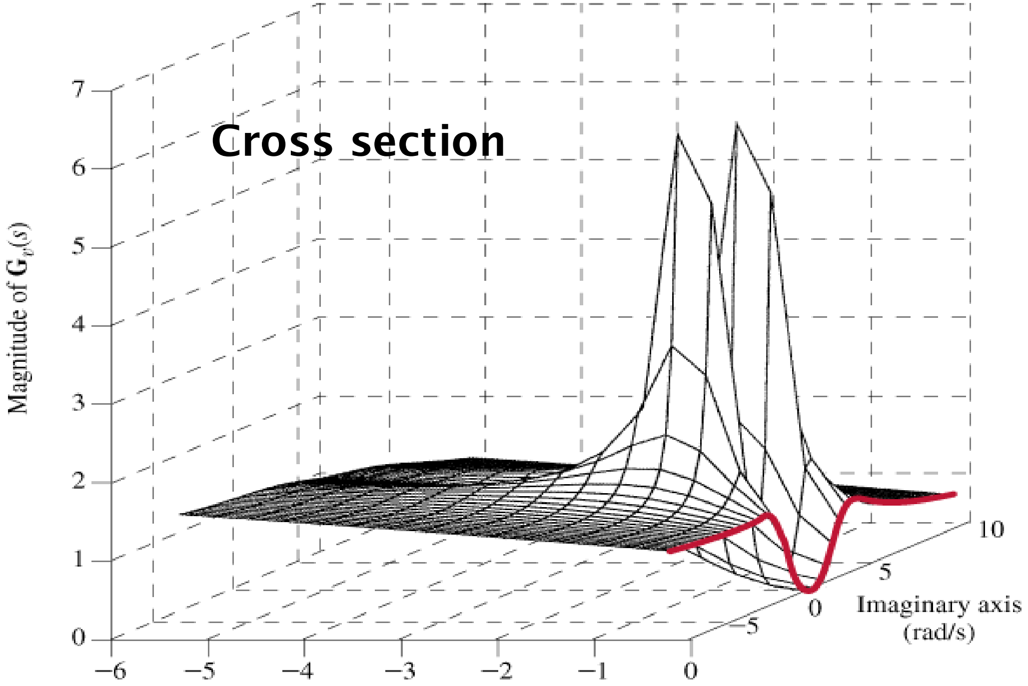
(a)



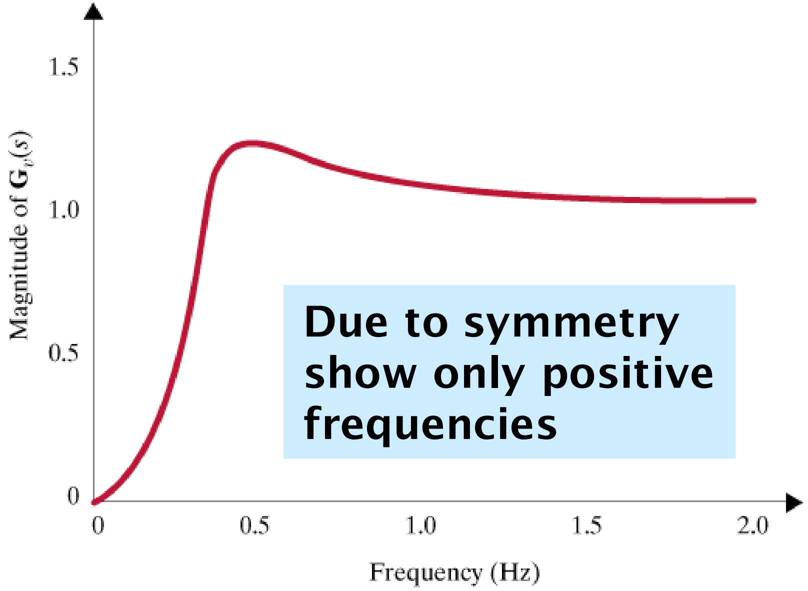
Cross section shown by Bode

(b)

GEAUX

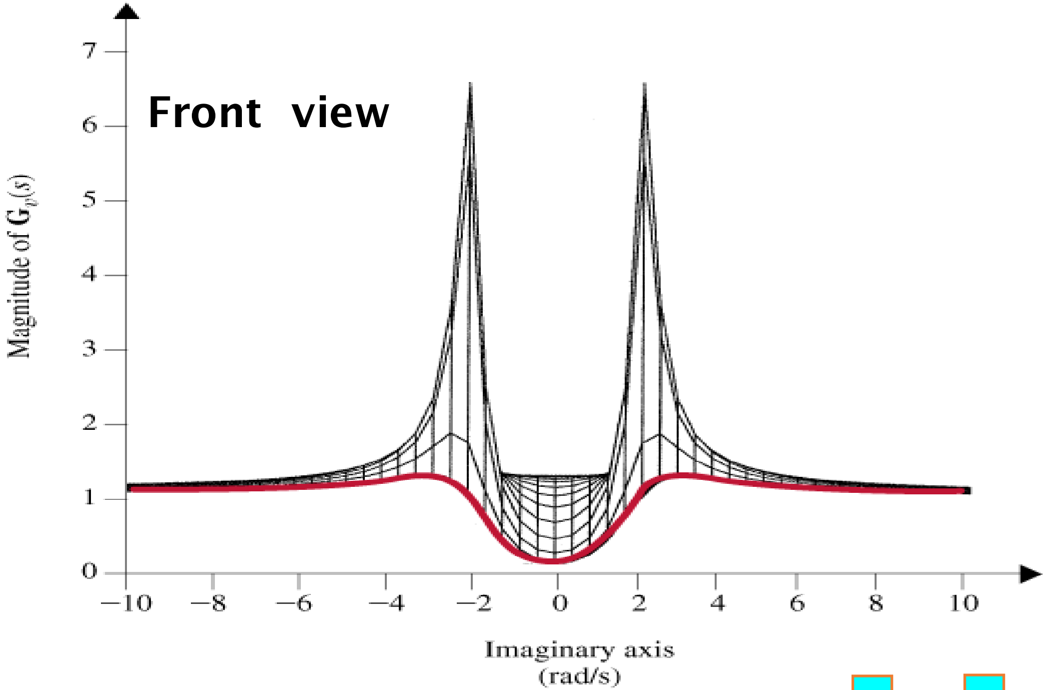


Cross section



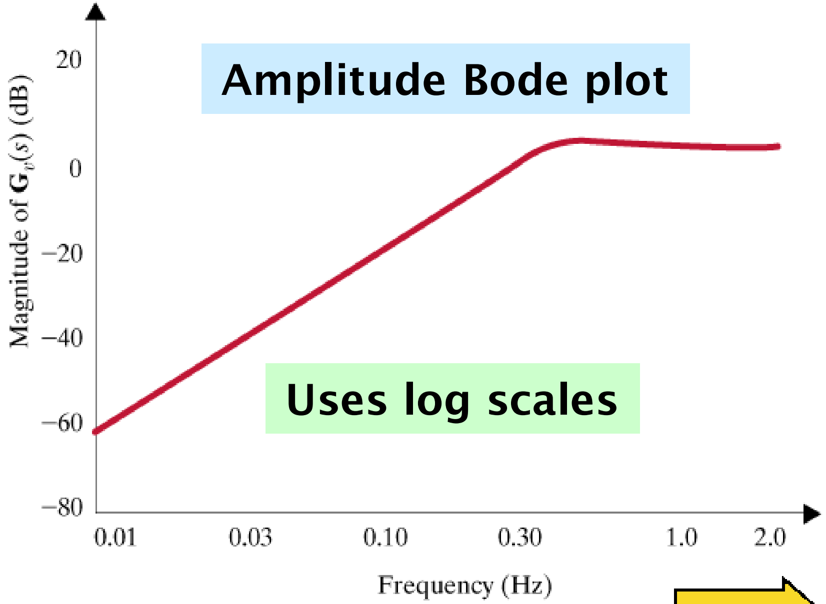
Due to symmetry show only positive frequencies

(e)



Front view

(d)



Amplitude Bode plot

Uses log scales

(f)

GEAUX

STEADY STATE RESPONSE

$Y(s) = H(s)X(s)$

Response when all initial conditions are zero

Laplace uses positive time functions. Even for sinusoids the response contains transitory terms

EXAMPLE

$$H(s) = \frac{1}{s+1}, X(s) = \frac{s}{s^2+\omega^2} \; (\Rightarrow x(t) = [\cos\omega t]u(t))$$

$$Y(s) = \frac{s}{(s+1)(s+j\omega)(s-j\omega)} = \frac{K_1}{s+1} + \frac{K_2}{s+j\omega} + \frac{K_2^*}{s-j\omega}$$

$$y(t) = \left(\boxed{K}_1 e^{-t} + \boxed{2}\,|K_2|\cos(\omega t + \angle K_2°)\right)u(t)$$

transient

Steady state response

If interested in the steady state response only, then don't determine residues associated with transient terms

$$\text{If } x(t) = X_M \cos(\omega_o t + \theta)u(t)$$

$$y_{ss}(t) = X_M\,|H(j\omega_o)|\cos(\omega_o t + \angle H(j\omega_o) + \theta)$$

For the general case

$$X_M \cos\omega t\,u(t) = \frac{X_M}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \Rightarrow X(s) = \frac{1}{2}\left(\frac{X_M}{s-j\omega_o} + \frac{X_M}{s+j\omega_o}\right)$$

$$Y(s) = H(s)\left[\frac{1}{2}\left(\frac{X_M}{s-j\omega_o} + \frac{X_M}{s+j\omega_o}\right)\right] = \frac{K_x}{s-j\omega_o} + \frac{K_x^*}{s+j\omega_o} + \text{transient terms}$$

$$K_x = (s-j\omega_o)Y(s)|_{s=j\omega_o} = \frac{1}{2}X_M H(j\omega_o)$$

$$y(t) = 2|K_x|\cos(\omega_o t + \angle K_2) + \text{transient terms}$$

$$y_{ss}(t) = X_M\,|H(j\omega_o)|\cos(\omega_o t + \angle H(j\omega_o))$$

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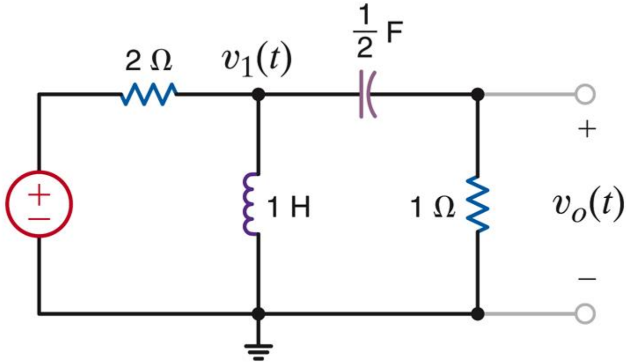
LEARNING EXAMPLE

Determine the steady state response

$v_i(t) = 10 \cos 2t\, u(t)$ V



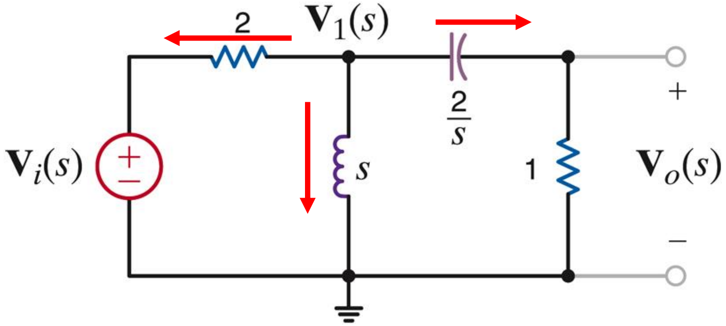
$$\omega_o = 2,\ X_M = 10$$

$$\text{If } x(t) = X_M \cos(\omega_o t + \theta) u(t)$$

$$y_{ss}(t) = X_M\, |H(j\omega_o)| \cos(\omega_o t + \angle H(j\omega_o) + \theta)$$

$$V_o(s) = \frac{s^2}{3s^2 + 4s + 4} V_i(s) \Rightarrow H(s) = \frac{s^2}{3s^2 + 4s + 4}$$

Transform the circuit to the Laplace domain. Assume all initial conditions are zero



$$\text{KCL@V}_1: \frac{V_1 - V_i}{2} + \frac{V_1}{s} + \frac{V_1}{\frac{2}{s} + 1} = 0$$

$$\text{Voltage divider}: V_o = \frac{1}{\frac{2}{s} + 1} V_1$$

$$H(j2) = \frac{(j2)^2}{3(j2)^2 + 4(j2) + 4} = 0.354\angle 45°$$

$$\therefore y_s(t) = 3.54\cos(2t + 45°)V$$

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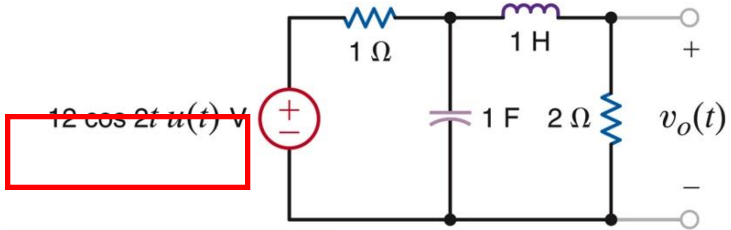
LEARNING EXTENSION

Determine $v_{oss}(t), t > 0$



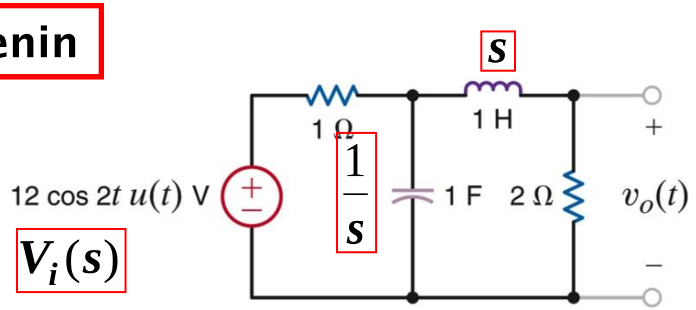
If $x(t) = X_M \cos(\omega_o t + \theta)u(t)$

$y_{ss}(t) = X_M \,|H(j\omega_o)|\cos(\omega_o t + \angle H(j\omega_o) + \theta)$

$\omega_o = 2,\ X_M = 12$

Transform circuit to Laplace domain.
Assume all initial conditions are zero

Thevenin



$$V_{OC}(s) = \frac{\frac{1}{s}}{1+\frac{1}{s}}V_i(s) = \frac{1}{s+1}V_i(s)$$

$$Z_{Th}(s) = s + \|1, \frac{1}{s}\| = s + \frac{1}{s+1} = \frac{s^2+s+1}{s+1}$$

$$V_o(s) = \frac{2}{2+Z_{Th}(s)}V_{OC}(s)$$

$$V_o(s) = \frac{2}{2+\frac{s^2+s+1}{s+1}} \times \frac{1}{s+1}V_i(s)$$

$$V_o(s) = \frac{2}{s^2+3s+3}V_i(s)$$

$H(s)$

$$H(j2) = \frac{2}{-4+6j+3} = \frac{2}{-1+6j} = \frac{2}{6.08\angle 99.46°}$$

$$v_{oss}(t) = 12 \times \frac{2}{6.08}\cos(2t - 99.46°)$$

APPLICATION LAPLACE

STOP

TWO – PORT NETWORKS

- A pair of terminals through which a current may enter or leave a network is known as a *port*.
- Two terminal devices or elements (such as resistors, capacitors, and inductors) results in one – port network.
- Most of the circuits we have dealt with so far are two – terminal or one – port circuits.